Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

- Tuletusreeglid

$$\dfrac{\begin{matrix}\overline{\gamma}^{(i)}\\ \vdots\\ \varphi\end{matrix}}{\gamma\to\varphi^{(i)}}\ \to\mathsf{I} \qquad \dfrac{\begin{matrix}\vdots & \vdots\\ \gamma\to\varphi & \gamma\end{matrix}}{\varphi}\ \to\mathsf{E}$$

$$\dfrac{\begin{matrix}\vdots & \vdots\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}\ \wedge\mathsf{I} \qquad \dfrac{\begin{matrix}\vdots\\ \varphi_1\wedge\varphi_2\end{matrix}}{\varphi_1}\ \wedge\mathsf{E}_\mathrm{L} \qquad \dfrac{\begin{matrix}\vdots\\ \varphi_1\wedge\varphi_2\end{matrix}}{\varphi_2}\ \wedge\mathsf{E}_\mathrm{R}$$

$$\dfrac{\begin{matrix}\vdots\\ \varphi_1\end{matrix}}{\varphi_1\vee\varphi_2}\ \vee\mathsf{I}_\mathrm{L} \qquad \dfrac{\begin{matrix}\vdots\\ \varphi_2\end{matrix}}{\varphi_1\vee\varphi_2}\ \vee\mathsf{I}_\mathrm{R} \qquad \dfrac{\begin{matrix} & \overline{\varphi_1}^{(i)} & \overline{\varphi_2}^{(i)}\\ \vdots & \vdots & \vdots\\ \varphi_1\vee\varphi_2 & \gamma & \gamma\end{matrix}}{\gamma^{(i)}}\ \vee\mathsf{E}$$

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

- Näide:

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{\varphi}^{(2)} \quad \dfrac{\overline{(\varphi\to\psi)\wedge(\varphi\to\rho)}^{(1)}}{\varphi\to\psi}\ \wedge\mathsf{E}}{\psi}\ \to\mathsf{E} \qquad \dfrac{\overline{\varphi}^{(2)} \quad \dfrac{\overline{(\varphi\to\psi)\wedge(\varphi\to\rho)}^{(1)}}{\varphi\to\rho}\ \wedge\mathsf{E}}{\rho}\ \to\mathsf{E}}{\psi\wedge\rho}\ \wedge\mathsf{I}}{\varphi\to\psi\wedge\rho^{(2)}}\ \to\mathsf{I}}{(\varphi\to\psi)\wedge(\varphi\to\rho)\to\varphi\to\psi\wedge\rho^{(1)}}\ \to\mathsf{I}$$

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

- Normaalkujud

$$
\frac{\dfrac{\overline{\varphi^{(1)}}}{\varphi\to\varphi^{(1)}}\,\to\mathsf{I} \qquad \dfrac{\dfrac{\overline{\varphi\to\varphi^{(2)}}}{\psi\to\varphi\to\varphi^{(3)}}\,\to\mathsf{I}}{(\varphi\to\varphi)\to\psi\to\varphi\to\varphi^{(2)}}\,\to\mathsf{I}}{\psi\to\varphi\to\varphi}\,\to\mathsf{E}
\qquad\qquad
\frac{\dfrac{\overline{\varphi^{(1)}}\;\to\mathsf{I}}{\varphi\to\varphi^{(1)}}\,\to\mathsf{I}}{\psi\to\varphi\to\varphi^{(2)}}\,\to\mathsf{I}
$$

- *Teoreem:* Iga tõestatava valemi korral leidub normaalkujuline tõestus.

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

- Normaliseerimisreeglid:

$$
\frac{\dfrac{\begin{matrix}\vdots\,\Sigma\\ \varphi_1\end{matrix}\quad\begin{matrix}\vdots\,\Pi\\ \varphi_2\end{matrix}}{\varphi_1\wedge\varphi_2}}{\varphi_1}
\quad\to\quad
\begin{matrix}\vdots\,\Sigma\\ \varphi_1\end{matrix}
\qquad\qquad
\frac{\dfrac{\begin{matrix}\vdots\,\Theta\\ \varphi_1\end{matrix}}{\varphi_1\vee\varphi_2}\quad\begin{matrix}\overline{\varphi_1}\\ \vdots\,\Sigma\\ \gamma\end{matrix}\quad\begin{matrix}\overline{\varphi_2}\\ \vdots\,\Pi\\ \gamma\end{matrix}}{\gamma}
\quad\to\quad
\begin{matrix}\vdots\,\Theta\\ \varphi_1\\ \vdots\,\Sigma\\ \gamma\end{matrix}
$$

$$
\frac{\begin{matrix}\vdots\,\Sigma\\ \psi\end{matrix}\quad\dfrac{\begin{matrix}\overline{\psi}\\ \vdots\,\Pi\\ \varphi\end{matrix}}{\psi\to\varphi}}{\varphi}
\quad\to\quad
\begin{matrix}\vdots\,\Sigma\\ \psi\\ \vdots\,\Pi\\ \varphi\end{matrix}
$$

Curry-Howard isomorfism

- *Teoreem:*
  (i) kui $\Gamma \vdash M : \varphi$, siis $|\Gamma| \vdash_{ND(\rightarrow)} \varphi$, kus $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$;
  (ii) kui $\Gamma \vdash_{ND(\rightarrow)} \varphi$, siis leidub term M selline et $\Delta \vdash M : \varphi$, kus $\Delta = \{x_\varphi : \varphi \mid \varphi \in \Gamma\}$.
- Curry-Howard vastavus

$\lambda \rightarrow$	$ND(\rightarrow)$
tüüp	valem
term	tõestus
reduktsioon	normaliseerimine

Paarid ja summad ($\lambda(\rightarrow \times +)$)

- *Tüüpimisreeglid:*

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M,N) : \sigma \times \tau} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathtt{fst}\ M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathtt{snd}\ M : \tau}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathtt{inl}\ M : \sigma + \tau} \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathtt{inr}\ M : \sigma + \tau}$$

$$\frac{\Gamma \vdash L : \sigma + \tau \quad \Gamma, \{x : \sigma\} \vdash M : \rho \quad \Gamma, \{y : \tau\} \vdash N : \rho}{\Gamma \vdash \mathtt{case}(L; x.M; y.N) : \rho}$$

Paarid ja variandid ($\lambda(\rightarrow\times+)$)

- *Reduktsioonireeglid:*

$$
\begin{array}{lcl}
\texttt{fst}\ (M,N) & \rightarrow & M \\
\texttt{snd}\ (M,N) & \rightarrow & N \\
 & & \\
\texttt{case}(\texttt{inl}\,L; x.M; y.N) & \rightarrow & M[L/x] \\
\texttt{case}(\texttt{inr}\,L; x.M; y.N) & \rightarrow & N[L/y]
\end{array}
$$

- Kehtib Curry-Howardi isomorfism $\lambda(\rightarrow\times+)$ ja $ND(\rightarrow\wedge\vee)$ vahel.

Polümorfne λ-arvutus (F)

- *Tüübid:* $\tau\ :=\ \alpha\ \mid\ (\tau\rightarrow\tau)\ \mid\ (\forall\alpha.\ \tau)$
- *Termid:* $M := x\ \mid\ (\lambda x{:}\tau.M)\ \mid\ (M\,M)\ \mid\ (\Lambda\alpha.M)\ \mid\ (M\,\tau)$
- *Tüüpimisreeglid:*

$$
\frac{\Gamma \Vdash M : \sigma}{\Gamma \Vdash (\Lambda\alpha.M) : \forall\alpha.\sigma}\ (\alpha\notin FV(\Gamma)) \qquad\qquad \frac{\Gamma \Vdash M : \forall\alpha.\sigma}{\Gamma \Vdash M\,\tau : \sigma[\tau/\alpha]}
$$

- β-*reduktsioon:*

$$
\begin{array}{rcl}
(\lambda x{:}\tau.M)\,N & \rightarrow_\beta & M[N/x] \\
(\Lambda\alpha.M)\,\tau & \rightarrow_\beta & M[\tau/\alpha]
\end{array}
$$

Polümorfne λ-arvutus (F)

- Näited:

$$\vdash \Lambda\alpha.\lambda x^{\forall\alpha.\alpha\to\alpha}.x(\alpha\to\alpha)(x\alpha) : \forall\alpha.(\forall\alpha.\alpha\to\alpha)\to\alpha\to\alpha$$
$$\vdash \Lambda\alpha.\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f(fx) : \forall\alpha.(\alpha\to\alpha)\to(\alpha\to\alpha)$$

- *Teoreem:* Süsteemi **F** on rangelt normaliseeruv.
- Kehtib Curry-Howardi isomorfism süsteemi **F** ja teist järku intuitsionistliku predikaatarvutuse $\{\forall,\to\}$-fragmendi vahel.
- *Järeldus:* Tüübi asustatavuse probleem süsteemis **F** ei ole lahenduv.

Polümorfne λ-arvutus (F)

- *Absurd:* $\bot \equiv \forall\alpha.\alpha$
- *Konjunktsioon:* $\sigma\times\tau \equiv \forall\alpha.((\sigma\to\tau\to\alpha)\to\alpha)$

$$\begin{array}{lcl} (P,Q) & \equiv & \Lambda\alpha.\lambda z^{\sigma\to\tau\to\alpha}.zPQ \\ \mathtt{fst}(M^{\sigma\times\tau}) & \equiv & M\,\sigma\,(\lambda x^{\sigma}.\lambda y^{\tau}.x) \\ \mathtt{snd}(M^{\sigma\times\tau}) & \equiv & M\,\tau\,(\lambda x^{\sigma}.\lambda y^{\tau}.x) \end{array}$$

- *Disjunktsioon:* $\sigma+\tau \equiv \forall\alpha.((\sigma\to\alpha)\to(\tau\to\alpha)\to\alpha)$

$$\begin{array}{lcl} \mathtt{inl}(M^{\sigma}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.uM \\ \mathtt{inr}(M^{\tau}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.vM \\ \mathtt{case}(L^{\sigma+\tau};x^{\sigma}.M^{\rho};y^{\tau}.N^{\rho}) & \equiv & L\,\rho\,(\lambda x^{\sigma}.M)\,(\lambda y^{\tau}.N) \end{array}$$

Polümorfne λ-arvutus (F)

- *Churchi numbrid:* $\omega \equiv \forall\alpha.(\alpha\to\alpha)\to\alpha\to\alpha$

$$
\begin{array}{lcl}
\underline{n} & \equiv & \Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f^n x \\
\mathtt{succ} & \equiv & \lambda n^{\omega}.\Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f(n\alpha f x)
\end{array}
$$

- NB! Leidub rangelt normaliseeruvaid terme, mis ei ole süsteemis **F** tüübitavad: $(\lambda zy.y(z\mathtt{I})(z\mathtt{K}))(\lambda x.xx)$
- *Teoreem:* Süsteemis **F** defineeritavate funktsioonide klass langeb kokku teist järku Peano aritmeetikas tõestatavalt rekursiivsete funktsioonide klassiga.
- *Teoreem:* Polümorfse λ-arvutuse tüübituletamine ja tüübikontroll on ekvivalentsed ja mittelahenduvad probleemid.