Tüübi tuletamine

- *Näide:*

$$
\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \cfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \qquad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

- *Tähistused:*
  - $S, S', \ldots$ (tüübi)substitutsioonid.
  - $\tau \succ \tau' \iff \exists S[\tau' = S(\tau)]$;
  - $\Gamma \succ \Gamma' \iff \exists S[\Gamma' \supseteq S(\Gamma)]$.
- Paar (Γ, τ) on termi M *printsipiaalne paar* parajasti siis, kui
  (i) $\Gamma \vdash M : \tau$;
  (ii) $\Gamma' \vdash M : \tau' \iff \Gamma \succ \Gamma' \wedge \tau \succ \tau'$.

  Kui M on kinnine term ($\Gamma = \emptyset$), siis τ on *printsipiaalne tüüp*.
- *Teoreem:* Kui term M on tüübitav, siis leidub talle printsipiaalne paar. See paar on unikaalne (tüübi)muutujate ümbernimetamise täpsuseni.

Tüübi tuletamine

- *Algoritm:*
  - Sisend: algkontekst Γ ja term M
  - Väljund: tüüp τ_M ja võrrandite süsteem E_M
    1. kui $M \equiv x$ ja $x \notin \mathrm{dom}(\Gamma)$, siis $\tau_M = \alpha_x$ ja $E_M = \emptyset$;
    2. kui $M \equiv x$ ja $x \in \mathrm{dom}(\Gamma)$, siis $\tau_M = \Gamma(x)$ ja $E_M = \emptyset$;
    3. kui $M \equiv PQ$, siis $\tau_M = \alpha$ ja $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \to \alpha\}$;
    4. kui $M \equiv \lambda x.P$, siis $\tau_M = \alpha_x \to \tau_P$ ja $E_M = E_P$
  - Lõpuks leitakse saadud võrrandisüsteemi lahendav *kõige üldisem unifikaator* U; kui seda ei leidu siis pole term tüübitav, vastasel korral on termi tüüp $U(\tau_M)$.

Tüübi tuletamine

- Näide:

$$\dfrac{\dfrac{\dfrac{\dfrac{(x:\alpha_x,\ \emptyset)\qquad \dfrac{(y:\alpha_y,\ \emptyset)\qquad (z:\alpha_z,\ \emptyset)}{(yz:\beta,\ \{\alpha_y=\alpha_z\to\beta\})}}{(x(yz):\gamma, \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda z.x(yz):\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda yz.x(yz):\alpha_y\to\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda xyz.x(yz):\alpha_x\to\alpha_y\to\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}$$

- Mittetüüdbitava termi näide:

$$\dfrac{\dfrac{(x:\alpha_x,\ \emptyset)\qquad (x:\alpha_x,\ \emptyset)}{(xx:\beta,\ \{\alpha_x=\alpha_x\to\beta\})}}{(\lambda x.xx:\alpha_x\to\beta,\ \{\alpha_x=\alpha_x\to\beta\})}$$

Hindley-Milner'i tüübisüsteem

- *Tüübid:* $\tau ::= \alpha \mid (\tau \to \tau)$
- *Tüübiskeemid:* $\sigma ::= \tau \mid \forall\alpha.\,\sigma$
- *Tüüpimisreeglid:*

$$\frac{}{\Gamma, x:\tau \vdash x:\tau} \qquad \frac{\Gamma \vdash M:\tau \quad \Gamma,\{x{:}\tau\} \vdash N:\sigma}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N:\sigma}$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma\to\tau} \qquad \frac{\Gamma \vdash M:\sigma\to\tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash MN:\tau}$$

$$\frac{\Gamma \vdash M:\sigma}{\Gamma \vdash M:\forall\alpha.\sigma}\ (\alpha\notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma \vdash M:\forall\alpha.\sigma}{\Gamma \vdash M:\sigma[\tau/\alpha]}$$

- *Reduktsioon:* $\texttt{let}\ x = M\ \texttt{in}\ N \to N[M/x]$

Hindley-Milner'i tüübisüsteem

- `Let`-seotud vs. λ-seotud muutujad

$$\Gamma = \{3{:}\texttt{Int},\ \texttt{True}{:}\texttt{Bool},\ (,){:}\forall\alpha\beta.\alpha\to\beta\to\alpha\times\beta\}$$

$$\Gamma \vdash \texttt{let}\ id = \lambda x.x\ \texttt{in}\ (id\,3,\ id\,\texttt{True}) : \texttt{Int}\times\texttt{Bool}$$

$$\Gamma \nvdash (\lambda id.(id\,3,\ id\,\texttt{True}))\ (\lambda x.x) : \texttt{Int}\times\texttt{Bool}$$

- Modifitseeritud tüüpimisreeglid:

$$\frac{}{\Gamma, x:\sigma \vdash x:\tau}\ (\sigma\succ\tau) \qquad \frac{\Gamma \vdash M:\tau \quad \Gamma,\{x{:}\forall\bar{\alpha}.\tau\} \vdash N:\sigma}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N:\sigma}\ (\bar{\alpha}\notin \mathrm{FV}(\Gamma))$$

Hindley-Milner'i tüübisüsteem

- Näide:

$$\dfrac{\dfrac{}{\Gamma,\{x{:}\alpha\}\vdash x:\alpha}}{\Gamma\vdash \lambda x.x:\alpha\to\alpha}\quad \dfrac{\dfrac{\overline{\Gamma'\vdash id:\mathtt{Int}\to\mathtt{Int}}\quad \overline{\Gamma'\vdash 3:\mathtt{Int}}}{\Gamma'\vdash id\,3:\mathtt{Int}}\quad \dfrac{\overline{\Gamma'\vdash id:\mathtt{B}\to\mathtt{B}}\quad \overline{\Gamma'\vdash \mathtt{T}:\mathtt{B}}}{\Gamma'\vdash id\,\mathtt{T}:\mathtt{B}}}{\Gamma,\{id{:}\forall\alpha.\alpha\to\alpha\}\vdash (id\,3,\ id\,\mathtt{T}):\mathtt{Int}\times\mathtt{B}}$$
$$\overline{\Gamma\vdash \mathtt{let}\ id=\lambda x.x\ \mathtt{in}\ (id\,3,\ id\,\mathtt{T}):\mathtt{Int}\times\mathtt{B}}$$

- Väga komplitseeritud tüübiga term:

$$\begin{array}{l}
\mathtt{let}\ pair=\lambda xyz.zxy\ \mathtt{in}\\
\ \mathtt{let}\ x_1=\lambda y.pair\,y\,y\ \mathtt{in}\\
\ \ \mathtt{let}\ x_2=\lambda y.x_1(x_1\,y)\ \mathtt{in}\\
\ \ \ \mathtt{let}\ x_3=\lambda y.x_2(x_2\,y)\ \mathtt{in}\\
\ \ \ \ \mathtt{let}\ x_4=\lambda y.x_3(x_3\,y)\ \mathtt{in}\\
\ \ \ \ \ \mathtt{let}\ x_5=\lambda y.x_4(x_4\,y)\ \mathtt{in}\\
\ \ \ \ \ \ x_5(\lambda y.y)
\end{array}$$