Andmete esitamine λ-arvutuses

- Baaskombinaatorid

$$\begin{array}{lcl} \mathbf{I} & \equiv & \lambda x.\, x \\ \mathbf{K} & \equiv & \lambda x\, y.\, x \\ \mathbf{S} & \equiv & \lambda f\, g\, x.\, f\, x\, (g\, x) \end{array}$$

- "Astendamine"

$$\begin{array}{lcl} E^0 E' & \equiv & E' \\ E^n E' & \equiv & \underbrace{E(E(\ldots(E}_{n\ \text{tükki}}\ E')\ldots)) \end{array}$$

- **NB!** $E^n (E\ E') \ \equiv\ E^{n+1} E' \ \equiv\ E\ (E^n E')$

Tõeväärtused

- Spetsifikatsioon

$$\begin{aligned}\textbf{not true} &= \textbf{false}\\ \textbf{not false} &= \textbf{true}\end{aligned}$$

- Definitsioon

$$\begin{aligned}\textbf{true} &\equiv \lambda xy.\ x \qquad (\equiv \quad \mathbf{K})\\ \textbf{false} &\equiv \lambda xy.\ y\\ \textbf{not} &\equiv \lambda t.\ t\ \textbf{false true}\end{aligned}$$

- Näide:

$$\begin{aligned}\textbf{not true} &\equiv (\lambda t.\ t\ \textbf{false true})\ \textbf{true}\\ &\longrightarrow \textbf{true false true}\\ &\equiv (\lambda x.\ \lambda y.\ x)\ \textbf{false true}\\ &\longrightarrow (\lambda y.\ \textbf{false})\ \textbf{true}\\ &\longrightarrow \textbf{false}\end{aligned}$$

Tingimuslause

- Spetsifikatsioon

$$\begin{aligned} \textbf{cond true } E_1\ E_2 &= E_1 \\ \textbf{cond false } E_1\ E_2 &= E_2 \end{aligned}$$

- Definitsioon

$$\textbf{cond} \equiv \lambda t\ x\ y.\ t\ x\ y$$

- Näide:

$$\begin{aligned} \textbf{cond false } E_1\ E_2 &\equiv (\lambda t\ x\ y.\ t\ x\ y)\ \textbf{false } E_1\ E_2 \\ &\Longrightarrow \textbf{false } E_1\ E_2 \\ &\equiv (\lambda x\ y.\ y)\ E_1\ E_2 \\ &\Longrightarrow E_2 \end{aligned}$$

Paarid ja ennikud

- Paarid

$$\begin{aligned}
\mathbf{fst} &\equiv \lambda p.\ p\ \mathbf{true} \\
\mathbf{snd} &\equiv \lambda p.\ p\ \mathbf{false} \\
(E_1, E_2) &\equiv \lambda f.\ f\ E_1\ E_2
\end{aligned}$$

- Ennikud

$$\begin{aligned}
(E_1, \ldots, E_n) &\equiv (E_1, (\ldots (E_{n-1}, E_n) \ldots)) \\
E \overset{n}{\downarrow} 1 &\equiv \mathbf{fst}\ E \\
E \overset{n}{\downarrow} 2 &\equiv \mathbf{fst}\ (\mathbf{snd}\ E) \\
&\ldots \\
E \overset{n}{\downarrow} i &\equiv \mathbf{fst}\ (\mathbf{snd}^{i-1} E) \\
&\ldots \\
E \overset{n}{\downarrow} n &\equiv \mathbf{snd}^{n-1}\ E
\end{aligned}$$

Naturaalarvud

- Standardnumbrid

$$
\begin{array}{lcll}
\ulcorner 0 \urcorner & \equiv & \lambda x.\ x & (\equiv \quad \mathbf{I}) \\
\ulcorner n{+}1 \urcorner & \equiv & (\mathbf{false}, \ulcorner n \urcorner) & \\
\mathbf{succ} & \equiv & \lambda n.\ (\mathbf{false}, n) & \\
\mathbf{pred} & \equiv & \lambda n.\ n\ \mathbf{false} & (\equiv \quad \mathbf{snd}) \\
\mathbf{iszero} & \equiv & \lambda n.\ n\ \mathbf{true} & (\equiv \quad \mathbf{fst})
\end{array}
$$

- Liitmine (?!)

$$
\mathbf{add} \quad = \quad \lambda x\ y.\ \mathbf{cond}(\mathbf{iszero}\ x)\ y\ (\mathbf{add}(\mathbf{pred}\ x)(\mathbf{succ}\ y))
$$

Naturaalarvud

- Church'i numbrid

$$\begin{array}{lcl}
\underline{n} & \equiv & \lambda f\ x.\ f^n\ x \\
\textbf{succ} & \equiv & \lambda n.\ \lambda f\ x.\ n\ f\ (f\ x) \\
\textbf{iszero} & \equiv & \lambda n.\ n\ (\lambda x.\ \textbf{false})\ \textbf{true} \\
\textbf{add} & \equiv & \lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x)
\end{array}$$

- Näide

$$\begin{array}{rcl}
\textbf{add}\ \underline{2}\ \underline{1} & \equiv & (\lambda m\ n.\ \lambda f\ x.\ m\ f\ (n\ f\ x))\ \underline{2}\ \underline{1} \\
& \Longrightarrow & \lambda f\ x.\ \underline{2}\ f\ (\underline{1}\ f\ x) \\
& \Longrightarrow & \lambda f\ x.\ f\ (f\ (\underline{1}\ f\ x)) \\
& \Longrightarrow & \lambda f\ x.\ f\ (f\ (f\ x)) \\
& \equiv & \underline{3}
\end{array}$$

- Korrutamine ja astendamine

$$\begin{array}{lcl}
\textbf{mul} & \equiv & \lambda m\ n.\ \lambda f\ x.\ m\ (n\ f)\ x \\
\textbf{exp} & \equiv & \lambda m\ n.\ \lambda f\ x.\ n\ m\ f\ x
\end{array}$$

Naturaalarvud

- Ühe lahutamine — abifunktsiooni spetsifikatsioon

$$
\begin{array}{lcl}
\textbf{prefn}\ f\ (\textbf{true}, x) & = & (\textbf{false}, x) \\
\textbf{prefn}\ f\ (\textbf{false}, x) & = & (\textbf{false}, f\ x) \\
(\textbf{prefn}\ f)^n\ (\textbf{false}, x) & = & (\textbf{false}, f^n\ x) \\
(\textbf{prefn}\ f)^n\ (\textbf{true}, x) & = & (\textbf{false}, f^{n-1}\ x)
\end{array}
$$

- Ühe lahutamine — definitsioon

$$
\begin{array}{lcl}
\textbf{prefn} & \equiv & \lambda f\ p.\ (\textbf{false}, (\textbf{cond}\ (\textbf{fst}\ p)\ (\textbf{snd}\ p)\ (f\ (\textbf{snd}\ p)))) \\
\textbf{pred} & \equiv & \lambda n.\ \lambda f\ x.\ \textbf{snd}\ (n\ (\textbf{prefn}\ f)\ (\textbf{true}, x))
\end{array}
$$

- Näide

$$
\begin{array}{lcl}
\textbf{pred}\ \underline{n}\ f\ x & = & \textbf{snd}\ (\underline{n}\ (\textbf{prefn}\ f)\ (\textbf{true}, x)) \\
 & = & \textbf{snd}\ ((\textbf{prefn}\ f)^n\ (\textbf{true}, x)) \\
 & = & \textbf{snd}\ (\textbf{false}, f^{n-1}\ x) \\
 & = & f^{n-1}\ x
\end{array}
$$

Listid

- Definitsioon

$$
\begin{array}{lcll}
\mathbf{nil} & \equiv & \lambda z.\, z & (\equiv \quad \mathbf{I}) \\
\mathbf{cons} & \equiv & \lambda x\ y.\, (\mathbf{false}, (x, y)) & \\
\mathbf{null} & \equiv & \lambda z.\, z\ \mathbf{true} & (\equiv \quad \mathbf{fst}) \\
\mathbf{hd} & \equiv & \lambda z.\, \mathbf{fst}\ (\mathbf{snd}\ z)) & \\
\mathbf{tl} & \equiv & \lambda z.\, \mathbf{snd}\ (\mathbf{snd}\ z)) &
\end{array}
$$

- Näide

$$
\begin{array}{rcl}
\mathbf{null\ nil} & \equiv & \mathbf{fst}\ (\lambda z.\, z) \\
 & \equiv & (\lambda p.\, p\ \mathbf{true})\ (\lambda z.\, z) \\
 & \Longrightarrow & \mathbf{true}
\end{array}
$$

Püsipunktid

- Püsipunktiteoreem:

$$\forall F.\ \exists X.\ X = F\,X$$

- Termi M nimetatakse püsipunkti kombinaatoriks kui

$$\forall F.\ M\,F = F\,(M\,F)$$

- Curry "paradoksaalne" kombinaator

$$\mathbf{Y} \equiv \lambda f.(\lambda x.f\,(x\,x))\,(\lambda x.f\,(x\,x))$$

- "Tugev" püsipunkti kombinaator

$$\mathbf{\Theta} \equiv (\lambda xy.\ y(xxy))\,(\lambda xy.\ y(xxy))$$

Püsipunktid

- Lemma: Olgu $G \equiv \lambda yf.f(yf) \qquad (\equiv \mathbf{SI})$

$$M \text{ on püsipunkti kombinaator} \quad \Longleftrightarrow \quad M = GM$$

- Tõestus:

($\Leftarrow$) Kui $M = GM$, siis:

$$\begin{array}{rcl} \forall F.\ M\,F & = & G\,M\,F \\ & \equiv & (\lambda yf.f(yf))\,M\,F \\ & = & F(M\,F) \end{array}$$

($\Rightarrow$) Kui M on püsipunkti kombinaator, siis:

$$\begin{array}{rcl} G\,M & = & \lambda f.f(Mf) \\ & = & \lambda f.Mf \\ & = & M \end{array}$$

Püsipunktid

- Lemma: Kõik jada

$$\begin{aligned} \mathbf{Y}^0 &\equiv \mathbf{Y} \\ \mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G \end{aligned}$$

  elemendid on püsipunkti kombinaatorid.

- Tõestus:

$$\begin{aligned} \mathbf{Y}^{n+1} &\equiv \mathbf{Y}^n\, G \\ &= G\, (\mathbf{Y}^n\, G) \\ &\equiv G\, \mathbf{Y}^{n+1} \end{aligned}$$

- NB! $\mathbf{Y}^1 \Longrightarrow \mathbf{\Theta}$

Rekursioon

- Lemma: Olgu $C \equiv C(f, \vec{x})$, siis
  - $\exists F.\ \forall \vec{N}.\ F\,\vec{N} = C(F, \vec{N})$
  - $\exists F.\ \forall \vec{N}.\ F\,\vec{N} \Longrightarrow C(F, \vec{N})$
- Tõestus: Võtame $F \equiv \mathbf{\Theta}(\lambda f \vec{x}. C(f, \vec{x}))$
- Standardnumbrite liitmine

$$\begin{aligned}
\mathbf{add} &= \lambda x\ y.\ \mathbf{cond}\,(\mathbf{iszero}\ x)\ y\ (\mathbf{add}(\mathbf{pred}\ x)(\mathbf{succ}\ y)) \\
\mathbf{add} &\equiv \mathbf{Y}\,(\lambda f\ x\ y.\ \mathbf{cond}\,(\mathbf{iszero}\ x)\ y\ (f\,(\mathbf{pred}\ x)(\mathbf{succ}\ y)))
\end{aligned}$$

Mitmekohalised funktsioonid

- "currymine"

$$\begin{aligned} \mathbf{curry}_n &\equiv \lambda f\ x_1\ \ldots\ x_n.\ f\ (x_1, \ldots, x_n) \\ \mathbf{uncurry}_n &\equiv \lambda f\ p.\ f\ (p \overset{n}{\downarrow} 1)\ \ldots\ (p \overset{n}{\downarrow} n) \end{aligned}$$

- NB! $\mathbf{curry}_n(\mathbf{uncurry}_n\ N) = N$ $\qquad$ $\mathbf{uncurry}_n(\mathbf{curry}_n\ M) = M$
- Üldistatud λ-abstraktsioon

$$\lambda(V_1, \ldots, V_n).\ E \equiv \mathbf{uncurry}_n\ (\lambda V_1\ \ldots\ V_n.\ E)$$

- Üldistatud β-konversioon

$$(\lambda(V_1, \ldots, V_n).\ E)(E_1, \ldots, E_n) \longrightarrow_\beta E[E_1 \ldots E_n / V_1 \ldots V_n]$$

Rekursioon

- Vastastikune rekursioon

$$
\begin{array}{rcl}
f_1 & = & F_1\ f_1 \ldots f_n \\
f_2 & = & F_2\ f_1 \ldots f_n \\
 & \ldots & \\
f_n & = & F_n\ f_1 \ldots f_n
\end{array}
$$

- Definitsioon

$$
\begin{array}{rcl}
f & \equiv & \mathbf{Y}(\lambda(f_1, \ldots, f_n).\ (F_1 f_1 \ldots f_n,\ \ldots,\ F_n f_1 \ldots f_n)) \\
f_1 & \equiv & f \overset{n}{\downarrow} 1 \\
f_2 & \equiv & f \overset{n}{\downarrow} 2 \\
 & \ldots & \\
f_n & \equiv & f \overset{n}{\downarrow} n
\end{array}
$$

Arvutatavus

- Church'i tees: Kõik arvutatavad funktsioonid on esitatavad λ-arvutuses!
- Lihtrekursiivsed funktsioonid:
  - **(i)** 0
  - **(ii)** $S(x) = x + 1$
  - **(iii)** $U_n^i(x_1, x_2, \ldots, x_n) = x_i$
  - **(iv)** $f(x_1, \ldots, x_n) = g(h_1(x_1, \ldots, x_n), \ldots, h_r(x_1, \ldots, x_n))$
  - **(v)** $$f(0, x_2, \ldots, x_n) = g(x_2, \ldots, x_n)$$ $$f(S(x_1), x_2, \ldots, x_n) = h(f(x_1, x_2, \ldots, x_n), x_2, \ldots, x_n)$$

Arvutatavus

- Osaliselt rekursiivsed funktsioonid

(vi) $f(x_1, \ldots, x_n) = \min \{ \ y \mid g(y, x_2, \ldots, x_n) = x_1 \ \}$

- Minimiseerimine

$$\begin{aligned} \mathbf{min}\ x\ f\ (x_1, \ldots, x_n) \quad &= \quad \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n)\ x_1)\ \ x \\ &\qquad\qquad (\mathbf{min}\ (\mathbf{succ}\ x)\ f\ (x_1, \ldots, x_n)) \end{aligned}$$

- Definitsioon

$$\begin{aligned} \mathbf{min} \quad &\equiv \quad \mathbf{Y}\ (\lambda m\, x\, f\ (x_1, \ldots, x_n).\ \mathbf{cond}\ (\mathbf{eq}\ (f(x, x_2, \ldots, x_n))\ x_1)\ \ x \\ &\qquad\qquad (m(\mathbf{succ}\ x)\ f\ (x_1, \ldots, x_n))) \end{aligned}$$