Naturaaldeduktsioon ($ND(\to \land \lor)$)

- Tuletusreeglid

$$\dfrac{\begin{array}{c}\overline{\gamma^{(i)}}\\ \vdots\\ \varphi\end{array}}{\gamma \to \varphi^{(i)}}\ \to\mathsf{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \gamma \to \varphi\end{array} \quad \begin{array}{c}\vdots\\ \gamma\end{array}}{\varphi}\ \to\mathsf{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ \varphi_1\end{array} \quad \begin{array}{c}\vdots\\ \varphi_2\end{array}}{\varphi_1 \land \varphi_2}\ \land\mathsf{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1 \land \varphi_2\end{array}}{\varphi_1}\ \land\mathsf{E}_\mathrm{L} \qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1 \land \varphi_2\end{array}}{\varphi_2}\ \land\mathsf{E}_\mathrm{R}$$

$$\dfrac{\begin{array}{c}\vdots\\ \varphi_1\end{array}}{\varphi_1 \lor \varphi_2}\ \lor\mathsf{I}_\mathrm{L} \qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_2\end{array}}{\varphi_1 \lor \varphi_2}\ \lor\mathsf{I}_\mathrm{R} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1 \lor \varphi_2\end{array} \quad \begin{array}{c}\overline{\varphi_1^{(i)}}\\ \vdots\\ \gamma\end{array} \quad \begin{array}{c}\overline{\varphi_2^{(i)}}\\ \vdots\\ \gamma\end{array}}{\gamma^{(i)}}\ \lor\mathsf{E}$$

Naturaaldeduktsioon ($ND(\rightarrow\wedge\vee)$)

- Näide:

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\varphi^{(2)}}
      \qquad
      \dfrac{\dfrac{}{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}}{\varphi\to\psi}\,\wedge\mathsf{E}
    }{\psi}\,\to\!\mathsf{E}
    \qquad
    \dfrac{
      \dfrac{}{\varphi^{(2)}}
      \qquad
      \dfrac{\dfrac{}{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}}{\varphi\to\rho}\,\wedge\mathsf{E}
    }{\rho}\,\to\!\mathsf{E}
  }{
    \dfrac{\psi\wedge\rho}{\varphi\to\psi\wedge\rho^{(2)}}\,\to\!\mathsf{I}
  }\,\wedge\mathsf{I}
}{(\varphi\to\psi)\wedge(\varphi\to\rho)\to\varphi\to\psi\wedge\rho^{(1)}}\,\to\!\mathsf{I}
$$

Naturaaldeduktsioon ($ND(\rightarrow\wedge\vee)$)

- Normaalkujud

$$
\dfrac{\dfrac{\overline{\varphi^{(1)}}}{\varphi\rightarrow\varphi^{(1)}}\;{\rightarrow}\mathsf{I}
\qquad
\dfrac{\dfrac{\overline{\varphi\rightarrow\varphi^{(2)}}}{\psi\rightarrow\varphi\rightarrow\varphi^{(3)}}\;{\rightarrow}\mathsf{I}}{(\varphi\rightarrow\varphi)\rightarrow\psi\rightarrow\varphi\rightarrow\varphi^{(2)}}\;{\rightarrow}\mathsf{I}}{\psi\rightarrow\varphi\rightarrow\varphi}\;{\rightarrow}\mathsf{E}
\qquad\qquad
\dfrac{\dfrac{\overline{\varphi^{(1)}}\;{\rightarrow}\mathsf{I}}{\varphi\rightarrow\varphi^{(1)}}\;{\rightarrow}\mathsf{I}}{\psi\rightarrow\varphi\rightarrow\varphi^{(2)}}\;{\rightarrow}\mathsf{I}
$$

- Teoreem: Iga tõestatava valemi korral leidub normaalkujuline tõestus.

Naturaaldeduktsioon ($ND(\rightarrow\wedge\vee)$)

- Normaliseerimisreeglid:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array} \qquad \begin{array}{c}\vdots\ \Pi\\ \varphi_2\end{array}}{\varphi_1 \wedge \varphi_2}}{\varphi_1} \quad\longrightarrow\quad \begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Theta\\ \varphi_1\end{array}}{\varphi_1 \vee \varphi_2} \qquad \begin{array}{c}\overline{\varphi_1}\\ \vdots\ \Sigma\\ \gamma\end{array} \qquad \begin{array}{c}\overline{\varphi_2}\\ \vdots\ \Pi\\ \gamma\end{array}}{\gamma} \quad\longrightarrow\quad \begin{array}{c}\vdots\ \Theta\\ \varphi_1\\ \vdots\ \Sigma\\ \gamma\end{array}$$

$$\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \psi\end{array} \qquad \dfrac{\begin{array}{c}\overline{\psi}\\ \vdots\ \Pi\\ \varphi\end{array}}{\psi \rightarrow \varphi}}{\varphi} \quad\longrightarrow\quad \begin{array}{c}\vdots\ \Sigma\\ \psi\\ \vdots\ \Pi\\ \varphi\end{array}$$

Curry-Howard isomorfism

- Teoreem:

  (i) kui $\Gamma \vdash M : \varphi$, siis $|\Gamma| \vdash_{ND(\rightarrow)} \varphi$, kus $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$;

  (ii) kui $\Gamma \vdash_{ND(\rightarrow)} \varphi$, siis leidub term M selline et $\Delta \vdash M : \varphi$, kus $\Delta = \{x_\varphi : \varphi \mid \varphi \in \Gamma\}$.

- Curry-Howard vastavus

$\lambda \rightarrow$	$ND(\rightarrow)$
tüüp	valem
tüübimuutuja	lausemuutuja
term	tõestus
termimuutuja	eeldus
tüübikonstruktor	loogiline konnektiiv
asustatavus	tõestatavus
reduktsioon	normaliseerimine

Paarid ja summad ($\lambda(\rightarrow\times+)$)

- Tüüpimisreeglid:

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M, N) : \sigma \times \tau} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathtt{fst}\ M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \mathtt{snd}\ M : \tau}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \mathtt{inl}\ M : \sigma + \tau} \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \mathtt{inr}\ M : \sigma + \tau}$$

$$\frac{\Gamma \vdash L : \sigma + \tau \quad \Gamma, \{x : \sigma\} \vdash M : \rho \quad \Gamma, \{y : \tau\} \vdash N : \rho}{\Gamma \vdash \mathtt{case}(L; x.M; y.N) : \rho}$$

Paarid ja variandid ($\lambda(\rightarrow \times +)$)

- Reduktsioonireeglid:

$$\begin{array}{lcl} \texttt{fst}\ (M, N) & \longrightarrow & M \\ \texttt{snd}\ (M, N) & \longrightarrow & N \\ \\ \texttt{case}(\texttt{inl}\, L; x.M; y.N) & \longrightarrow & M[L/x] \\ \texttt{case}(\texttt{inr}\, L; x.M; y.N) & \longrightarrow & N[L/y] \end{array}$$

- Kehtib Curry-Howardi isomorfism $\lambda(\rightarrow \times +)$ ja $ND(\rightarrow \wedge \vee)$ vahel.

Polümorfne λ-arvutus (F)

- Tüübid: $\tau \;:=\; \alpha \;\mid\; (\tau \to \tau) \;\mid\; (\forall \alpha.\ \tau)$
- Termid: $M := \; x \;\mid\; (\lambda x{:}\tau.M) \;\mid\; (M\ M) \;\mid\; (\Lambda\alpha.M) \;\mid\; (M\ \tau)$
- Tüüpimisreeglid:

$$\frac{\Gamma \Vdash M : \sigma}{\Gamma \Vdash (\Lambda\alpha.M) : \forall\alpha.\sigma}\ (\alpha \notin \mathbf{FV}(\Gamma)) \qquad \frac{\Gamma \Vdash M : \forall\alpha.\sigma}{\Gamma \Vdash M\,\tau : \sigma[\tau/\alpha]}$$

- β-reduktsioon:

$$\begin{aligned} (\lambda x{:}\tau.M)\,N &\longrightarrow_\beta M[N/x] \\ (\Lambda\alpha.M)\,\tau &\longrightarrow_\beta M[\tau/\alpha] \end{aligned}$$

Polümorfne λ-arvutus (F)

- Näited:

$$\vdash \Lambda\alpha.\lambda x^{\forall\alpha.\alpha\to\alpha}.x(\alpha\to\alpha)(x\alpha) : \forall\alpha.(\forall\alpha.\alpha\to\alpha)\to\alpha\to\alpha$$

$$\vdash \Lambda\alpha.\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f(fx) : \forall\alpha.(\alpha\to\alpha)\to(\alpha\to\alpha)$$

- Teoreem: Süsteemi **F** on rangelt normaliseeruv.
- Kehtib Curry-Howardi isomorfism süsteemi **F** ja teist järku intuitsionistliku predikaatarvutuse $\{\forall,\to\}$-fragmendi vahel.
- Järeldus: Tüübi asustatavuse probleem süsteemis **F** ei ole lahenduv.

Polümorfne λ-arvutus (F)

- Absurd: $\bot \equiv \forall\alpha.\alpha$
- Konjunktsioon: $\sigma \times \tau \equiv \forall\alpha.((\sigma \to \tau \to \alpha) \to \alpha)$

$$\begin{array}{lcl}
(P, Q) & \equiv & \Lambda\alpha.\lambda z^{\sigma\to\tau\to\alpha}.zPQ \\
\mathtt{fst}(M^{\sigma\times\tau}) & \equiv & M\,\sigma\,(\lambda x^{\sigma}.\lambda y^{\tau}.x) \\
\mathtt{snd}(M^{\sigma\times\tau}) & \equiv & M\,\tau\,(\lambda x^{\sigma}.\lambda y^{\tau}.x)
\end{array}$$

- Disjunktsioon: $\sigma + \tau \equiv \forall\alpha.((\sigma \to \alpha) \to (\tau \to \alpha) \to \alpha)$

$$\begin{array}{lcl}
\mathtt{inl}(M^{\sigma}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.uM \\
\mathtt{inr}(M^{\tau}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.vM \\
\mathtt{case}(L^{\sigma+\tau}; x^{\sigma}.M^{\rho}; y^{\tau}.N^{\rho}) & \equiv & L\,\rho\,(\lambda x^{\sigma}.M)\,(\lambda y^{\tau}.N)
\end{array}$$

Polümorfne λ-arvutus (F)

- Churchi numbrid: $\omega \equiv \forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$

$$\begin{array}{lcl} \underline{n} & \equiv & \Lambda\alpha.\lambda f^{\alpha\to\alpha} x^{\alpha}.f^{n} x \\ \mathtt{succ} & \equiv & \lambda n^{\omega}.\Lambda\alpha.\lambda f^{\alpha\to\alpha} x^{\alpha}.f(n\alpha f x) \end{array}$$

- NB! Leidub rangelt normaliseeruvaid terme, mis ei ole süsteemis **F** tüübitavad: $(\lambda zy.y(z\mathtt{I})(z\mathtt{K}))(\lambda x.xx)$
- Teoreem: Süsteemis **F** defineeritavate funktsioonide klass langeb kokku teist järku Peano aritmeetikas tõestatavalt rekursiivsete funktsioonide klassiga.
- Teoreem: Polümorfse λ-arvutuse tüübituletamine ja tüübikontroll on ekvivalentsed ja mittelahenduvad probleemid.