Tüübi tuletamine

- Näide:

$$\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}}}}$$

Võrrandid:

$$\begin{array}{lll} \tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\ \tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8 \end{array}$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

- Tähistused:
  - $S, S', \ldots$ (tüübi)substitutsioonid.
  - $\tau \succ \tau' \iff \exists S\,[\tau' = S(\tau)]$;
  - $\Gamma \succ \Gamma' \iff \exists S\,[\Gamma' \supseteq S(\Gamma)]$.
- Paar (Γ, τ) on termi M printsipiaalne paar parajasti siis, kui

  (i) $\Gamma \vdash M : \tau$;

  (ii) $\Gamma' \vdash M : \tau' \iff \Gamma \succ \Gamma' \ \wedge\ \tau \succ \tau'$.

  Kui M on kinnine term ($\Gamma = \emptyset$), siis τ on printsipiaalne tüüp.
- Teoreem: Kui term M on tüübitav, siis leidub talle printsipiaalne paar. See paar on unikaalne (tüübi)muutujate ümbernimetamise täpsuseni.

Tüübi tuletamine

- Algoritm:
  - Sisend: algkontekst Γ ja term M
  - Väljund: tüüp τ_M ja võrrandite süsteem E_M
    1. kui $M \equiv x$ ja $x \notin \mathrm{dom}(\Gamma)$, siis $\tau_M = \alpha_x$ ja $E_M = \emptyset$;
    2. kui $M \equiv x$ ja $x \in \mathrm{dom}(\Gamma)$, siis $\tau_M = \Gamma(x)$ ja $E_M = \emptyset$;
    3. kui $M \equiv PQ$, siis $\tau_M = \alpha$ ja $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \to \alpha\}$;
    4. kui $M \equiv \lambda x.P$, siis $\tau_M = \alpha_x \to \tau_P$ ja $E_M = E_P$
  - Lõpuks leitakse saadud võrrandisüsteemi lahendav kõige üldisem unifikaator U; kui seda ei leidu siis pole term tüübitav, vastasel korral on termi tüüp $U(\tau_M)$.

Tüübi tuletamine

- Näide:

$$\dfrac{(x : \alpha_x,\ \emptyset) \quad \dfrac{(y : \alpha_y,\ \emptyset) \quad (z : \alpha_z,\ \emptyset)}{(yz : \beta,\ \{\alpha_y = \alpha_z \to \beta\})}}{\dfrac{(x(yz) : \gamma, \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}{\dfrac{(\lambda z.x(yz) : \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}{\dfrac{(\lambda yz.x(yz) : \alpha_y \to \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}{(\lambda xyz.x(yz) : \alpha_x \to \alpha_y \to \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}}}}$$

- Mittetüüdbitava termi näide:

$$\dfrac{\dfrac{(x : \alpha_x,\ \emptyset) \quad (x : \alpha_x,\ \emptyset)}{(xx : \beta,\ \{\alpha_x = \alpha_x \to \beta\})}}{(\lambda x.xx : \alpha_x \to \beta,\ \{\alpha_x = \alpha_x \to \beta\})}$$

Hindley-Milner'i tüübisüsteem

- Tüübid: $\tau \;:=\; \alpha \;\mid\; (\tau \to \tau)$
- Tüübiskeemid: $\sigma \;:=\; \tau \;\mid\; \forall\alpha.\,\sigma$
- Tüüpimisreeglid:

$$\frac{}{\Gamma, x:\tau \vdash x:\tau} \qquad \frac{\Gamma \vdash M:\tau \quad \Gamma,\{x{:}\tau\} \vdash N:\sigma}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N:\sigma}$$

$$\frac{\Gamma, x:\sigma \vdash M:\tau}{\Gamma \vdash \lambda x.M:\sigma \to \tau} \qquad \frac{\Gamma \vdash M:\sigma \to \tau \quad \Gamma \vdash N:\sigma}{\Gamma \vdash M\,N:\tau}$$

$$\frac{\Gamma \vdash M:\sigma}{\Gamma \vdash M:\forall\alpha.\sigma}\ (\alpha \notin \mathbf{FV}(\Gamma)) \qquad \frac{\Gamma \vdash M:\forall\alpha.\sigma}{\Gamma \vdash M:\sigma[\tau/\alpha]}$$

- Reduktsioon: $\texttt{let}\ x = M\ \texttt{in}\ N \longrightarrow N[M/x]$

Hindley-Milner'i tüübisüsteem

- `Let`-seotud vs. λ-seotud muutujad

$$\Gamma = \{3{:}\texttt{Int},\ \texttt{True}{:}\texttt{Bool},\ (,){:}\forall\alpha\beta.\alpha \to \beta \to \alpha \times \beta\}$$

$$\Gamma \vdash \texttt{let}\ id = \lambda x.x\ \texttt{in}\ (id\,3,\ id\,\texttt{True}) : \texttt{Int} \times \texttt{Bool}$$

$$\Gamma \nvdash (\lambda id.(id\,3,\ id\,\texttt{True}))\ (\lambda x.x) : \texttt{Int} \times \texttt{Bool}$$

- Modifitseeritud tüüpimisreeglid:

$$\frac{}{\Gamma, x : \sigma \vdash x : \tau}\ (\sigma \succ \tau) \qquad \frac{\Gamma \vdash M : \tau \quad \Gamma, \{x{:}\forall\bar{\alpha}.\tau\} \vdash N : \sigma}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N : \sigma}\ (\bar{\alpha} \notin \mathbf{FV}(\Gamma))$$

Hindley-Milner'i tüübisüsteem

- Näide:

$$
\dfrac{
  \dfrac{\dfrac{}{\Gamma, \{x{:}\alpha\} \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}
  \quad
  \dfrac{
    \dfrac{\dfrac{}{\Gamma' \vdash \mathit{id} : \mathtt{Int} \to \mathtt{Int}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{Int}}}{\Gamma' \vdash \mathit{id}\,3 : \mathtt{Int}}
    \quad
    \dfrac{\dfrac{}{\Gamma' \vdash \mathit{id} : \mathtt{B} \to \mathtt{B}} \quad \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}}{\Gamma' \vdash \mathit{id}\,\mathtt{T} : \mathtt{B}}
  }{\Gamma, \{\mathit{id}{:}\forall\alpha.\alpha \to \alpha\} \vdash (\mathit{id}\,3,\ \mathit{id}\,\mathtt{T}) : \mathtt{Int} \times \mathtt{B}}
}{\Gamma \vdash \mathtt{let}\ \mathit{id} = \lambda x.x\ \mathtt{in}\ (\mathit{id}\,3,\ \mathit{id}\,\mathtt{T}) : \mathtt{Int} \times \mathtt{B}}
$$

- Väga komplitseeritud tüübiga term:

$$
\begin{array}{l}
\mathtt{let}\ \mathit{pair} = \lambda xyz.z\,x\,y\ \mathtt{in} \\
\ \mathtt{let}\ x_1 = \lambda y.\mathit{pair}\,y\,y\ \mathtt{in} \\
\ \ \mathtt{let}\ x_2 = \lambda y.x_1(x_1\,y)\ \mathtt{in} \\
\ \ \ \mathtt{let}\ x_3 = \lambda y.x_2(x_2\,y)\ \mathtt{in} \\
\ \ \ \ \mathtt{let}\ x_4 = \lambda y.x_3(x_3\,y)\ \mathtt{in} \\
\ \ \ \ \ \mathtt{let}\ x_5 = \lambda y.x_4(x_4\,y)\ \mathtt{in} \\
\ \ \ \ \ \ x_5(\lambda y.y)
\end{array}
$$