Tüübi tuletamine

Näide

$$\dfrac{\dfrac{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}$$

Võrrandid:

$$\tau_0 = \tau_1 \to \tau_2 \qquad \tau_2 = \tau_3 \to \tau_4 \qquad \tau_4 = \tau_5 \to \tau_6$$
$$\tau_1 = \tau_7 \to \tau_6 \qquad \tau_3 = \tau_8 \to \tau_7 \qquad \tau_5 = \tau_8$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \Gamma \vdash yz : \tau_7}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}$$

Võrrandid:

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$$\tau_1 = \tau_7 \to \tau_6 \qquad \tau_3 = \tau_8 \to \tau_7 \qquad \tau_5 = \tau_8$$

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Tüübi tuletamine

Näide

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}$$

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Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$\dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \qquad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}$$

$$\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \Gamma \vdash yz : \tau_7}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}$$

$$\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}$$

$$\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}$$

$$\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}$$

Võrrandid:

$$\tau_0 = \tau_1 \to \tau_2 \qquad \tau_2 = \tau_3 \to \tau_4 \qquad \tau_4 = \tau_5 \to \tau_6$$

$$\tau_1 = \tau_7 \to \tau_6 \qquad \tau_3 = \tau_8 \to \tau_7 \qquad \tau_5 = \tau_8$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \qquad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}}}}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

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$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
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Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}}}}
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Võrrandid:

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\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
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\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Tähistused

- $S, S', \ldots$ (tüübi)substitutsioonid.
- $\tau \succ \tau' \iff \exists S\,[\tau' = S(\tau)]$;
- $\Gamma \succ \Gamma' \iff \exists S\,[\Gamma' \supseteq S(\Gamma)]$.

Definitsioon

Paar (Γ, τ) on termi M printsipiaalne paar parajasti siis, kui

(i) $\Gamma \vdash M : \tau$;

(ii) $\Gamma' \vdash M : \tau' \iff \Gamma \succ \Gamma' \ \wedge \ \tau \succ \tau'$.

Kui M on kinnine term ($\Gamma = \emptyset$), siis τ on printsipiaalne tüüp.

Teoreem

Kui term M on tüübitav, siis leidub talle printsipiaalne paar. See paar on unikaalne (tüübi)muutujate ümbernimetamise täpsuseni.

Tüübi tuletamine

Algoritm

- Sisend: algkontekst Γ ja term M
- Väljund: tüüp τ_M ja võrrandite süsteem E_M
  - kui $M \equiv x$ ja $x \notin \mathrm{dom}(\Gamma)$, siis $\tau_M = \alpha_x$ ja $E_M = \emptyset$;
  - kui $M \equiv x$ ja $x \in \mathrm{dom}(\Gamma)$, siis $\tau_M = \Gamma(x)$ ja $E_M = \emptyset$;
  - kui $M \equiv PQ$, siis $\tau_M = \alpha$ ja $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \to \alpha\}$;
  - kui $M \equiv \lambda x.P$, siis $\tau_M = \alpha_x \to \tau_P$ ja $E_M = E_P$
- Lõpuks leitakse saadud võrrandisüsteemi lahendav kõige üldisem unifikaator U; kui seda ei leidu siis pole term tüübitav, vastasel korral on termi tüüp $U(\tau_M)$.

Tüübi tuletamine

Näide

$$\dfrac{\dfrac{\dfrac{\dfrac{(x:\alpha_x,\ \emptyset)\quad \dfrac{(y:\alpha_y,\ \emptyset)\quad (z:\alpha_z,\ \emptyset)}{(yz:\beta,\ \{\alpha_y=\alpha_z\to\beta\})}}{(x(yz):\gamma,\{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda z.x(yz):\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda yz.x(yz):\alpha_y\to\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}}{(\lambda xyz.x(yz):\alpha_x\to\alpha_y\to\alpha_z\to\gamma,\ \{\alpha_y=\alpha_z\to\beta,\ \alpha_x=\beta\to\gamma\})}$$

Mittetüüdbitava termi näide

$$\dfrac{\dfrac{(x:\alpha_x,\ \emptyset)\quad (x:\alpha_x,\ \emptyset)}{(xx:\beta,\ \{\alpha_x=\alpha_x\to\beta\})}}{(\lambda x.xx:\alpha_x\to\beta,\ \{\alpha_x=\alpha_x\to\beta\})}$$

Hindley-Milner'i tüübisüsteem

- *Tüübid:* $\tau \;:=\; \alpha \;\mid\; (\tau \to \tau)$
- *Tüübiskeemid:* $\sigma \;:=\; \tau \;\mid\; \forall\alpha.\,\sigma$
- *Tüüpimisreeglid:*

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma \vdash M : \tau \quad \Gamma, \{x{:}\tau\} \vdash N : \sigma}{\Gamma \vdash \mathtt{let}\ x = M\ \mathtt{in}\ N : \sigma}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau} \qquad \frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M\,N : \tau}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \forall\alpha.\sigma}\ (\alpha \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma \vdash M : \forall\alpha.\sigma}{\Gamma \vdash M : \sigma[\tau/\alpha]}$$

- *Reduktsioon:* $\mathtt{let}\ x = M\ \mathtt{in}\ N \longrightarrow N[M/x]$

Hindley-Milner'i tüübisüsteem

Let-seotud vs. λ-seotud muutujad

$$\Gamma = \{3{:}\texttt{Int},\ \texttt{True}{:}\texttt{Bool},\ (,){:}\forall\alpha\beta.\alpha\to\beta\to\alpha\times\beta\}$$

$$\Gamma \vdash \texttt{let}\ id = \lambda x.x\ \texttt{in}\ (id\,3,\ id\,\texttt{True}) : \texttt{Int}\times\texttt{Bool}$$

$$\Gamma \nvdash (\lambda id.(id\,3,\ id\,\texttt{True}))\ (\lambda x.x) : \texttt{Int}\times\texttt{Bool}$$

Modifitseeritud tüüpimisreeglid

$$\frac{}{\Gamma, x:\sigma \vdash x:\tau}\ (\sigma \succ \tau)$$

$$\frac{\Gamma \vdash M:\tau \quad \Gamma, \{x{:}\forall\bar{\alpha}.\tau\} \vdash N:\sigma}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N:\sigma}\ (\bar{\alpha}\notin \mathrm{FV}(\Gamma))$$

Hindley-Milner'i tüübisüsteem

Näide

$$
\dfrac{\dfrac{}{\Gamma, \{x{:}\alpha\} \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}
\quad
\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \mathrm{I} \to \mathrm{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathrm{I}}}{\Gamma' \vdash id\, 3 : \mathrm{I}} \quad \dfrac{\dfrac{}{\Gamma' \vdash id : \mathrm{B} \to \mathrm{B}} \quad \dfrac{}{\Gamma' \vdash \mathrm{T} : \mathrm{B}}}{\Gamma' \vdash id\, \mathrm{T} : \mathrm{B}}}{\Gamma, \{id{:}\forall\alpha.\alpha \to \alpha\} \vdash (id\, 3,\ id\, \mathrm{T}) : \mathrm{I} \times \mathrm{B}}
$$
$$
\overline{\Gamma \vdash \mathtt{let}\ id = \lambda x.x\ \mathtt{in}\ (id\, 3,\ id\, \mathrm{T}) : \mathrm{I} \times \mathrm{B}}
$$

Väga komplitseeritud tüübiga term

$$
\begin{array}{l}
\mathtt{let}\ pair = \lambda xyz.z\,x\,y\ \mathtt{in} \\
\quad \mathtt{let}\ x_1 = \lambda y.pair\,y\,y\ \mathtt{in} \\
\quad\quad \mathtt{let}\ x_2 = \lambda y.x_1(x_1\,y)\ \mathtt{in} \\
\quad\quad\quad \mathtt{let}\ x_3 = \lambda y.x_2(x_2\,y)\ \mathtt{in} \\
\quad\quad\quad\quad \mathtt{let}\ x_4 = \lambda y.x_3(x_3\,y)\ \mathtt{in} \\
\quad\quad\quad\quad\quad \mathtt{let}\ x_5 = \lambda y.x_4(x_4\,y)\ \mathtt{in} \\
\quad\quad\quad\quad\quad\quad x_5(\lambda y.y)
\end{array}
$$