Tüübid

Lihtsalt tüübitud λ-arvutus

Tüübid

Olgu U (loenduv) tüübimuutujate hulk; lihtsate tüüpide hulk Π on defineeritud järgneva grammatikaga:

$$\Pi \;:=\; U \;\mid\; (\Pi \to \Pi)$$

Tüübimuutujaid tähistame $\alpha, \beta, \ldots \in U$; tüüpe $\sigma, \tau, \ldots \in \Pi$.

Kontekstid

Muutujast ja tüübist koosnevate paaride jada

$$\Gamma = \{x_1 : \tau_1,\ \ldots,\ x_n : \tau_n\}$$

Kontekstide hulka tähistame C ning kontektsi Γ doomenit $\mathrm{dom}(\Gamma) = \{x_1,\ \ldots,\ x_n\}$.

Lihtsalt tüübitud λ-arvutus à la Curry

Tüüpimine

Tüüpimisrelatsioon $\vdash \subseteq C \times \Lambda \times \Pi$ on defineeritud järgmiste tuletusreeglitega:

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

$$\frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M\ N : \tau}$$

kus $x \notin \mathrm{dom}(\Gamma)$.

Lihtsalt tüübitud λ-arvutus à la Curry

Definitsioon

Lihtsalt tüübitud λ-arvutus à la Curry ($\lambda\!\rightarrow$ à la Curry) on kolmik $(\Lambda, \Pi, \vdash)$.

Näiteid

$$\vdash \lambda x.x : \sigma \to \sigma$$
$$\vdash \lambda xy.x : \sigma \to \tau \to \sigma$$
$$\vdash \lambda xyz.x\,z\,(y\,z) : (\sigma \to \tau \to \rho) \to (\sigma \to \tau) \to \sigma \to \rho$$

Lihtsalt tüübitud λ-arvutus à la Curry

Näide

$$\dfrac{\Gamma \vdash x : \sigma \to \tau \qquad \dfrac{\Gamma \vdash y : \rho \to \sigma \quad \Gamma \vdash z : \rho}{\Gamma \vdash yz : \sigma}}{\dfrac{\Gamma \vdash x(yz) : \tau}{\dfrac{\Gamma_2 \vdash \lambda z.x(yz) : \rho \to \tau}{\dfrac{\Gamma_1 \vdash \lambda yz.x(yz) : (\rho \to \sigma) \to \rho \to \tau}{\vdash \lambda xyz.x(yz) : (\sigma \to \tau) \to (\rho \to \sigma) \to \rho \to \tau}}}}$$

kus

$$\begin{array}{lcl} \Gamma_1 & = & \{x{:}\sigma \to \tau\} \\ \Gamma_2 & = & \{x{:}\sigma \to \tau,\ y{:}\rho \to \sigma\} \\ \Gamma & = & \{x{:}\sigma \to \tau,\ y{:}\rho \to \sigma,\ z{:}\rho\} \end{array}$$

Lihtsalt tüübitud λ-arvutus à la Curry

Moodustamislemma

$$
\begin{array}{lcl}
\Gamma \vdash x : \sigma & \Rightarrow & x : \sigma \in \Gamma \\
\Gamma \vdash M\,N : \sigma & \Rightarrow & \exists \tau\ [\Gamma \vdash M : \tau \to \sigma\ \wedge\ \Gamma \vdash N : \tau] \\
\Gamma \vdash \lambda x.M : \sigma & \Rightarrow & \exists \tau, \rho\ [\Gamma, x : \tau \vdash M : \rho\ \wedge\ \sigma = \tau \to \rho]
\end{array}
$$

Subjektreduktsioon

Kui $\Gamma \vdash M : \sigma$ ja $M \Longrightarrow N$, siis $\Gamma \vdash N : \sigma$.

Teoreem

Relatsioon $\Longrightarrow$ on Church-Rosser.

Lihtsalt tüübitud λ-arvutus à la Church

Pseudotermid

Pseudotermide hulk Λ_Π on defineeritud grammatikaga:

$$\Lambda_\Pi \;:=\; V \;\mid\; (\lambda V : \Pi.\Lambda_\Pi) \;\mid\; (\Lambda_\Pi\, \Lambda_\Pi)$$

Tüüpimine

Tüüpimisrelatsioon $\Vdash \subseteq C \times \Lambda_\Pi \times \Pi$

$$\frac{}{\Gamma, x : \tau \Vdash x : \tau} \qquad \frac{\Gamma, x : \sigma \Vdash M : \tau}{\Gamma \Vdash \lambda x : \sigma.M : \sigma \to \tau}$$

$$\frac{\Gamma \Vdash M : \sigma \to \tau \quad \Gamma \Vdash N : \sigma}{\Gamma \Vdash M\; N : \tau}$$

Lihtsalt tüübitud λ-arvutus à la Church

Definitsioon

Lihtsalt tüübitud λ-arvutus à la Church ($\lambda\!\rightarrow$ à la Church) on kolmik $(\Lambda_\Pi, \Pi, \Vdash)$.

Näiteid

$$\Vdash \lambda x : \sigma.\ x : \sigma \rightarrow \sigma$$
$$\Vdash \lambda x : \sigma.\ \lambda y : \tau.\ x : \sigma \rightarrow \tau \rightarrow \sigma$$
$$\begin{aligned}\Vdash \lambda x : \sigma \rightarrow \tau \rightarrow \rho.\ \lambda y : \sigma \rightarrow \tau.\ \lambda z : \sigma.\ x\,z\,(y\,z)& \\ : (\sigma \rightarrow \tau \rightarrow \rho) \rightarrow (\sigma \rightarrow \tau) \rightarrow \sigma \rightarrow \rho&\end{aligned}$$

Lihtsalt tüübitud λ-arvutus à la Church

Moodustamislemma

$$\begin{array}{lcl}
\Gamma \Vdash x : \sigma & \Rightarrow & x : \sigma \in \Gamma \\
\Gamma \Vdash M\,N : \sigma & \Rightarrow & \exists \tau \; [\Gamma \Vdash M : \tau \to \sigma \;\wedge\; \Gamma \Vdash N : \tau] \\
\Gamma \Vdash \lambda x{:}\tau.M : \sigma & \Rightarrow & \exists \rho \; [\Gamma, x : \tau \Vdash M : \rho \;\wedge\; \sigma = \tau \to \rho]
\end{array}$$

Subjektreduktsioon

Kui $\Gamma \Vdash M : \sigma$ ja $M \Longrightarrow N$, siis $\Gamma \Vdash N : \sigma$.

Teoreem

Relatsioon $\Longrightarrow$ on Curch-Rosser.

Tüüpide unikaalsus

- Kui $\Gamma \Vdash M : \sigma$ ja $\Gamma \Vdash M : \tau$, siis $\sigma = \tau$.
- Kui $\Gamma \Vdash M : \sigma$ ja $\Gamma \Vdash N : \tau$ ja $M = N$, siis $\sigma = \tau$.

Church vs. Curry

Definitsioon

Tüüpide kustutamise funktsioon $|\cdot| : \Lambda_\Pi \to \Lambda$

$$
\begin{array}{rcl}
|x| & = & x \\
|M\ N| & = & |M|\ |N| \\
|\lambda x{:}\sigma.M| & = & \lambda x.|M|
\end{array}
$$

Teoreem

Olgu $M, N \in \Lambda_\Pi$

- kui $M \longrightarrow N$, siis $|M| \longrightarrow |N|$;
- kui $\Gamma \Vdash M : \sigma$, siis $\Gamma \vdash |M| : \sigma$.

Church vs. Curry

Teoreem

Olgu $M, N \in \Lambda$

- kui $M \longrightarrow N$, siis iga $M' \in \Lambda_\Pi$ mille korral $|M'| = M$ leidub $N' \in \Lambda_\Pi$ nii et $|N'| = N$ ja $M' \longrightarrow N'$;
- kui $\Gamma \vdash M{:}\sigma$, siis leidub $M' \in \Lambda_\Pi$ nii et $|M'| = M$ ja $\Gamma \Vdash M'{:}\sigma$.

Väljendusvõimsus

Definitsioon

Olgu $N = (\alpha \to \alpha) \to (\alpha \to \alpha)$. Funktsioon $f : \mathbf{N}^m \to \mathbf{N}$ on $\lambda\!\to$-defineeritav kui leidub $F \in \Lambda$ nii et $\vdash F : N \to \ldots \to N$ ($m + 1$ korda) ja $F\,\underline{n_1} \ldots \underline{n_m} = \underline{f(n_1, \ldots, n_m)}$.

Teoreem

$\lambda\!\to$-defineeritavad funktsioonid on täpselt laiendatud polünoomid.

Väljendusvõimsus

Laiendatud polünoomid

(1) *projektsioonid*: $U_i^m(n_1, \ldots, n_m) = n_i$ $(1 \leqslant i \leqslant m)$;

(2) *konstantfunktsioonid*: $k(n) = k$;

(3) *märgifunktsioon*: $\mathsf{sg}(0) = 0$ ja $\mathsf{sg}(m+1) = 1$;

(4) *liitmine*:
$$(f+g)(n_1, \ldots, n_k, m_1, \ldots, m_l) = f(n_1, \ldots, n_k) + g(m_1, \ldots, m_l);$$

(5) *korrutamine*:
$$(f \cdot g)(n_1, \ldots, n_k, m_1, \ldots, m_l) = f(n_1, \ldots, n_k) \cdot g(m_1, \ldots, m_l).$$

Lahenduvus

Definitsioon

Termi M nimetatakse ($\lambda\rightarrow$) tüübitavaks parajasti siis, kui leiduvad Γ ja τ mille korral $\Gamma \vdash M : \tau$.

Lemma

$\lambda\rightarrow$ tüübitavate termide klass on lahenduv (so. leidub algoritm, mis tuvastab kas antud term on tüübitav).

Lahenduvus

Nõrgalt normaliseeruvuse teoreem

Iga $\lambda\!\rightarrow$ tüübitav term omab nii β- kui ka $\beta\eta$-normaalkuju.

Tugevalt normaliseeruvuse teoreem

Iga $\lambda\!\rightarrow$ tüübitavast termist lähtuv β- ($\beta\eta$-) reduktsioonijada on lõplik.

Teoreem

$\lambda\!\rightarrow$ tüübitavate termide β- ($\beta\eta$-) võrduse kontrollimiseks leidub otsustusprotseduur (so. algoritm, mis saades 2 tüübitud termi P ja Q, tuvastab kas $P = Q$).

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Tuletusreeglid

$$\dfrac{\begin{array}{c}\overline{\gamma^{(i)}}\\ \vdots\\ \varphi\end{array}}{\gamma\to\varphi^{(i)}}\ \to\mathsf{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \gamma\to\varphi\end{array}\quad \begin{array}{c}\vdots\\ \gamma\end{array}}{\varphi}\ \to\mathsf{E}$$

$$\dfrac{\begin{array}{c}\vdots\\ \varphi_1\end{array}\quad \begin{array}{c}\vdots\\ \varphi_2\end{array}}{\varphi_1\wedge\varphi_2}\ \wedge\mathsf{I} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1\wedge\varphi_2\end{array}}{\varphi_1}\ \wedge\mathsf{E}_{\mathrm{L}} \qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1\wedge\varphi_2\end{array}}{\varphi_2}\ \wedge\mathsf{E}_{\mathrm{R}}$$

$$\dfrac{\begin{array}{c}\vdots\\ \varphi_1\end{array}}{\varphi_1\vee\varphi_2}\ \vee\mathsf{I}_{\mathrm{L}} \qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_2\end{array}}{\varphi_1\vee\varphi_2}\ \vee\mathsf{I}_{\mathrm{R}} \qquad\qquad \dfrac{\begin{array}{c}\vdots\\ \varphi_1\vee\varphi_2\end{array}\quad \begin{array}{c}\overline{\varphi_1^{(i)}}\\ \vdots\\ \gamma\end{array}\quad \begin{array}{c}\overline{\varphi_2^{(i)}}\\ \vdots\\ \gamma\end{array}}{\gamma^{(i)}}\ \vee\mathsf{E}$$

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Näide

$$
\dfrac{
\dfrac{
\dfrac{}{\varphi^{(2)}}
\quad
\dfrac{
\dfrac{}{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}
}{\varphi\to\psi}\ \wedge\mathsf{E}
}{\psi}\ \to\mathsf{E}
\qquad
\dfrac{
\dfrac{}{\varphi^{(2)}}
\quad
\dfrac{
\dfrac{}{(\varphi\to\psi)\wedge(\varphi\to\rho)^{(1)}}
}{\varphi\to\rho}\ \wedge\mathsf{E}
}{\rho}\ \to\mathsf{E}
}{
\dfrac{
\dfrac{\psi\wedge\rho}{\varphi\to\psi\wedge\rho^{(2)}}\ \to\mathsf{I}
}{(\varphi\to\psi)\wedge(\varphi\to\rho)\to\varphi\to\psi\wedge\rho^{(1)}}\ \to\mathsf{I}
}\ \wedge\mathsf{I}
$$

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Normaalkujud

$$
\dfrac{
\dfrac{\overline{\varphi^{(1)}}}{\varphi\to\varphi^{(1)}}\,{\to}\mathrm{I}
\qquad
\dfrac{\dfrac{\overline{\varphi\to\varphi^{(2)}}}{\psi\to\varphi\to\varphi^{(3)}}\,{\to}\mathrm{I}}{(\varphi\to\varphi)\to\psi\to\varphi\to\varphi^{(2)}}\,{\to}\mathrm{I}
}{\psi\to\varphi\to\varphi}\,{\to}\mathrm{E}
\qquad\qquad
\dfrac{\dfrac{\overline{\varphi^{(1)}}\,{\to}\mathrm{I}}{\varphi\to\varphi^{(1)}}\,{\to}\mathrm{I}}{\psi\to\varphi\to\varphi^{(2)}}\,{\to}\mathrm{I}
$$

Teoreem

Iga tõestatava valemi korral leidub normaalkujuline tõestus.

Naturaaldeduktsioon ($ND(\to\wedge\vee)$)

Normaliseerimisreeglid

$$
\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array}\qquad\begin{array}{c}\vdots\ \Pi\\ \varphi_2\end{array}}{\varphi_1\wedge\varphi_2}}{\varphi_1}
\quad\longrightarrow\quad
\begin{array}{c}\vdots\ \Sigma\\ \varphi_1\end{array}
\qquad\qquad
\dfrac{\dfrac{\begin{array}{c}\vdots\ \Theta\\ \varphi_1\end{array}}{\varphi_1\vee\varphi_2}\qquad\begin{array}{c}\overline{\varphi_1}\\ \vdots\ \Sigma\\ \gamma\end{array}\qquad\begin{array}{c}\overline{\varphi_2}\\ \vdots\ \Pi\\ \gamma\end{array}}{\gamma}
\quad\longrightarrow\quad
\begin{array}{c}\vdots\ \Theta\\ \varphi_1\\ \vdots\ \Sigma\\ \gamma\end{array}
$$

$$
\dfrac{\begin{array}{c}\vdots\ \Sigma\\ \psi\end{array}\qquad\dfrac{\begin{array}{c}\overline{\psi}\\ \vdots\ \Pi\\ \varphi\end{array}}{\psi\to\varphi}}{\varphi}
\quad\longrightarrow\quad
\begin{array}{c}\vdots\ \Sigma\\ \psi\\ \vdots\ \Pi\\ \varphi\end{array}
$$

Curry-Howard isomorfism

Teoreem

(i) Kui $\Gamma \vdash M : \varphi$, siis $|\Gamma| \vdash_{ND(\to)} \varphi$, kus $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$.

(ii) Kui $\Gamma \vdash_{ND(\to)} \varphi$, siis leidub term M selline et $\Delta \vdash M : \varphi$, kus $\Delta = \{x_\varphi : \varphi \mid \varphi \in \Gamma\}$.

Curry-Howard vastavus

$\lambda \to$	$ND(\to)$
tüüp	valem
tüübimuutuja	lausemuutuja
term	tõestus
termimuutuja	eeldus
tüübikonstruktor	loogiline konnektiiv
asustatavus	tõestatavus
reduktsioon	normaliseerimine

Paarid ja summad ($\lambda(\rightarrow \times +)$)

Tüüpimisreeglid

$$\frac{\Gamma \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M, N) : \sigma \times \tau} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \texttt{fst}\ M : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \times \tau}{\Gamma \vdash \texttt{snd}\ M : \tau}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash \texttt{inl}\ M : \sigma + \tau} \qquad \frac{\Gamma \vdash M : \tau}{\Gamma \vdash \texttt{inr}\ M : \sigma + \tau}$$

$$\frac{\Gamma \vdash L : \sigma + \tau \quad \Gamma, \{x : \sigma\} \vdash M : \rho \quad \Gamma, \{y : \tau\} \vdash N : \rho}{\Gamma \vdash \mathsf{case}(L; x.M; y.N) : \rho}$$

Paarid ja variandid ($\lambda(\to \times +)$)

Reduktsioonireeglid

$$
\begin{array}{lcl}
\mathtt{fst}\ (M, N) & \longrightarrow & M \\
\mathtt{snd}\ (M, N) & \longrightarrow & N \\
\\
\mathtt{case}(\mathtt{inl}\, L; x.M; y.N) & \longrightarrow & M[L/x] \\
\mathtt{case}(\mathtt{inr}\, L; x.M; y.N) & \longrightarrow & N[L/y]
\end{array}
$$

NB!

Kehtib Curry-Howardi isomorfism $\lambda(\to \times +)$ ja $ND(\to \wedge \vee)$ vahel.

Polümorfne λ-arvutus (F)

Tüübid

$$\tau \;:=\; \alpha \;\mid\; (\tau \to \tau) \;\mid\; (\forall \alpha.\ \tau)$$

Termid

$$M \;:=\; x \;\mid\; (\lambda x{:}\tau.M) \;\mid\; (M\ M) \;\mid\; (\Lambda\alpha.M) \;\mid\; (M\ \tau)$$

Tüüpimisreeglid

$$\frac{\Gamma \Vdash M : \sigma}{\Gamma \Vdash (\Lambda\alpha.M) : \forall\alpha.\sigma}\ (\alpha \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma \Vdash M : \forall\alpha.\sigma}{\Gamma \Vdash M\,\tau : \sigma[\tau/\alpha]}$$

β-reduktsioon

$$\begin{aligned} (\lambda x{:}\tau.M)\,N &\longrightarrow_\beta M[N/x] \\ (\Lambda\alpha.M)\,\tau &\longrightarrow_\beta M[\tau/\alpha] \end{aligned}$$

Polümorfne λ-arvutus (F)

Näited

$$\vdash \Lambda\alpha.\lambda x^{\forall\alpha.\alpha\to\alpha}.x(\alpha\to\alpha)(x\alpha) : \forall\alpha.(\forall\alpha.\alpha\to\alpha)\to\alpha\to\alpha$$
$$\vdash \Lambda\alpha.\lambda f^{\alpha\to\alpha}.\lambda x^{\alpha}.f(fx) : \forall\alpha.(\alpha\to\alpha)\to(\alpha\to\alpha)$$

Teoreem

Süsteem **F** on rangelt normaliseeruv.

NB!

Kehtib Curry-Howardi isomorfism süsteemi **F** ja teist järku intuitsionistliku predikaatarvutuse $\{\forall,\to\}$-fragmendi vahel.

Järeldus

Tüübi asustatavuse probleem süsteemis **F** ei ole lahenduv.

Polümorfne λ-arvutus (F)

- *Absurd:* $\bot \equiv \forall\alpha.\alpha$
- *Konjunktsioon:* $\sigma \times \tau \equiv \forall\alpha.((\sigma \to \tau \to \alpha) \to \alpha)$

$$
\begin{array}{lcl}
(P, Q) & \equiv & \Lambda\alpha.\lambda z^{\sigma\to\tau\to\alpha}.zPQ \\
\mathtt{fst}(M^{\sigma\times\tau}) & \equiv & M\,\sigma\,(\lambda x^{\sigma}.\lambda y^{\tau}.x) \\
\mathtt{snd}(M^{\sigma\times\tau}) & \equiv & M\,\tau\,(\lambda x^{\sigma}.\lambda y^{\tau}.y)
\end{array}
$$

- *Disjunktsioon:* $\sigma + \tau \equiv \forall\alpha.((\sigma \to \alpha) \to (\tau \to \alpha) \to \alpha)$

$$
\begin{array}{lcl}
\mathtt{inl}(M^{\sigma}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.uM \\
\mathtt{inr}(M^{\tau}) & \equiv & \Lambda\alpha.\lambda u^{\sigma\to\alpha}.\lambda v^{\tau\to\alpha}.vM \\
\mathtt{case}(L^{\sigma+\tau}; x^{\sigma}.M^{\rho}; y^{\tau}.N^{\rho}) & \equiv & L\,\rho\,(\lambda x^{\sigma}.M)\,(\lambda y^{\tau}.N)
\end{array}
$$

- *Churchi numbrid:* $\omega \equiv \forall\alpha.(\alpha \to \alpha) \to \alpha \to \alpha$

$$
\begin{array}{lcl}
\underline{n} & \equiv & \Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f^{n}x \\
\mathtt{succ} & \equiv & \lambda n^{\omega}.\Lambda\alpha.\lambda f^{\alpha\to\alpha}x^{\alpha}.f(n\alpha f x)
\end{array}
$$

Polümorfne λ-arvutus ($\mathbf{F}$)

NB!

Leidub rangelt normaliseeruvaid terme, mis ei ole süsteemis $\mathbf{F}$ tüübitavad

$$(\lambda zy.y(z\mathtt{I})(z\mathtt{K}))(\lambda x.xx)$$

Teoreem

Süsteemis $\mathbf{F}$ defineeritavate funktsioonide klass langeb kokku teist järku Peano aritmeetikas tõestatavalt rekursiivsete funktsioonide klassiga.

Teoreem

Polümorfse λ-arvutuse tüübituletamine ja tüübikontroll on ekvivalentsed ja mittelahenduvad probleemid.

Tüübi tuletamine

Näide

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\cfrac{\cfrac{\cfrac{\cfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \Gamma \vdash yz : \tau_7}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \qquad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}
}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}
}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}
}{\vdash \lambda xyz.x(yz) : \tau_0}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}}}}$$

Võrrandid:

$$\begin{array}{lll} \tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\ \tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8 \end{array}$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}}{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}}{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}}{\vdash \lambda xyz.x(yz) : \tau_0}$$

Võrrandid:

$$\begin{array}{lll} \tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\ \tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8 \end{array}$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Näide

$$
\dfrac{\Gamma \vdash x : \tau_7 \to \tau_6 \qquad \dfrac{\Gamma \vdash y : \tau_8 \to \tau_7 \quad \Gamma \vdash z : \tau_8}{\Gamma \vdash yz : \tau_7}}{\dfrac{\{x{:}\tau_1, y{:}\tau_3, z{:}\tau_5\} \vdash x(yz) : \tau_6}{\dfrac{\{x{:}\tau_1, y{:}\tau_3\} \vdash \lambda z.x(yz) : \tau_4}{\dfrac{\{x{:}\tau_1\} \vdash \lambda yz.x(yz) : \tau_2}{\vdash \lambda xyz.x(yz) : \tau_0}}}}
$$

Võrrandid:

$$
\begin{array}{lll}
\tau_0 = \tau_1 \to \tau_2 & \tau_2 = \tau_3 \to \tau_4 & \tau_4 = \tau_5 \to \tau_6 \\
\tau_1 = \tau_7 \to \tau_6 & \tau_3 = \tau_8 \to \tau_7 & \tau_5 = \tau_8
\end{array}
$$

Lahend: $\tau_0 = (\tau_7 \to \tau_6) \to (\tau_8 \to \tau_7) \to \tau_8 \to \tau_6$

Tüübi tuletamine

Tähistused

- $S, S', \ldots$ (tüübi)substitutsioonid.
- $\tau \succ \tau' \iff \exists S\, [\tau' = S(\tau)]$;
- $\Gamma \succ \Gamma' \iff \exists S\, [\Gamma' \supseteq S(\Gamma)]$.

Definitsioon

Paar (Γ, τ) on termi M printsipiaalne paar parajasti siis, kui

(i) $\Gamma \vdash M : \tau$;

(ii) $\Gamma' \vdash M : \tau' \iff \Gamma \succ \Gamma' \;\wedge\; \tau \succ \tau'$.

Kui M on kinnine term ($\Gamma = \emptyset$), siis τ on printsipiaalne tüüp.

Teoreem

Kui term M on tüübitav, siis leidub talle printsipiaalne paar. See paar on unikaalne (tüübi)muutujate ümbernimetamise täpsuseni.

Tüübi tuletamine

Algoritm

- Sisend: algkontekst Γ ja term M
- Väljund: tüüp τ_M ja võrrandite süsteem E_M
  - kui $M \equiv x$ ja $x \notin \mathrm{dom}(\Gamma)$, siis $\tau_M = \alpha_x$ ja $E_M = \emptyset$;
  - kui $M \equiv x$ ja $x \in \mathrm{dom}(\Gamma)$, siis $\tau_M = \Gamma(x)$ ja $E_M = \emptyset$;
  - kui $M \equiv PQ$, siis $\tau_M = \alpha$ ja
    $E_M = E_P \cup E_Q \cup \{\tau_P = \tau_Q \to \alpha\}$;
  - kui $M \equiv \lambda x.P$, siis $\tau_M = \alpha_x \to \tau_P$ ja $E_M = E_P$
- Lõpuks leitakse saadud võrrandisüsteemi lahendav kõige üldisem unifikaator U; kui seda ei leidu siis pole term tüübitav, vastasel korral on termi tüüp $U(\tau_M)$.

Tüübi tuletamine

Näide

$$
\dfrac{\dfrac{(x : \alpha_x,\ \emptyset) \quad \dfrac{(y : \alpha_y,\ \emptyset) \quad (z : \alpha_z,\ \emptyset)}{(yz : \beta,\ \{\alpha_y = \alpha_z \to \beta\})}}{(x(yz) : \gamma, \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}}{\dfrac{(\lambda z.x(yz) : \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}{\dfrac{(\lambda yz.x(yz) : \alpha_y \to \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}{(\lambda xyz.x(yz) : \alpha_x \to \alpha_y \to \alpha_z \to \gamma,\ \{\alpha_y = \alpha_z \to \beta,\ \alpha_x = \beta \to \gamma\})}}}
$$

Mittetüübitava termi näide

$$
\dfrac{\dfrac{(x : \alpha_x,\ \emptyset) \quad (x : \alpha_x,\ \emptyset)}{(xx : \beta,\ \{\alpha_x = \alpha_x \to \beta\})}}{(\lambda x.xx : \alpha_x \to \beta,\ \{\alpha_x = \alpha_x \to \beta\})}
$$

Hindley-Milner'i tüübisüsteem

- *Tüübid:* $\tau \;:=\; \alpha \;\mid\; (\tau \to \tau)$
- *Tüübiskeemid:* $\sigma \;:=\; \tau \;\mid\; \forall\alpha.\, \sigma$
- *Tüüpimisreeglid:*

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma} \qquad \frac{\Gamma \vdash M : \sigma \quad \Gamma, \{x{:}\sigma\} \vdash N : \tau}{\Gamma \vdash \mathtt{let}\ x = M\ \mathtt{in}\ N : \tau}$$

$$\frac{\Gamma, x : \tau_1 \vdash M : \tau_0}{\Gamma \vdash \lambda x.M : \tau_1 \to \tau_0} \qquad \frac{\Gamma \vdash M : \tau_1 \to \tau_0 \quad \Gamma \vdash N : \tau_1}{\Gamma \vdash M\ N : \tau_0}$$

$$\frac{\Gamma \vdash M : \sigma}{\Gamma \vdash M : \forall\alpha.\sigma}\ (\alpha \notin \mathrm{FV}(\Gamma)) \qquad \frac{\Gamma \vdash M : \forall\alpha.\sigma}{\Gamma \vdash M : \sigma[\tau/\alpha]}$$

- *Reduktsioon:* $\mathtt{let}\ x = M\ \mathtt{in}\ N \longrightarrow N[M/x]$

Hindley-Milner'i tüübisüsteem

Let-seotud vs. λ-seotud muutujad

$$\Gamma = \{3{:}\texttt{Int},\ \texttt{True}{:}\texttt{Bool},\ (,){:}\forall\alpha\beta.\alpha \to \beta \to \alpha \times \beta\}$$

$$\Gamma \vdash \texttt{let}\ id = \lambda x.x\ \texttt{in}\ (id\,3,\ id\,\texttt{True}) : \texttt{Int} \times \texttt{Bool}$$

$$\Gamma \nvdash (\lambda id.(id\,3,\ id\,\texttt{True}))\ (\lambda x.x) : \texttt{Int} \times \texttt{Bool}$$

Modifitseeritud tüüpimisreeglid

$$\frac{}{\Gamma, x : \sigma \vdash x : \tau}\ (\sigma \succ \tau)$$

$$\frac{\Gamma \vdash M : \tau_1 \quad \Gamma, \{x{:}\forall\bar{\alpha}.\tau_1\} \vdash N : \tau_0}{\Gamma \vdash \texttt{let}\ x = M\ \texttt{in}\ N : \tau_0}\ (\bar{\alpha} \notin \mathrm{FV}(\Gamma))$$

Hindley-Milner'i tüübisüsteem

Näide

$$\dfrac{\dfrac{}{\Gamma, \{x{:}\alpha\} \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha} \quad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash \mathit{id} : \mathtt{I} \to \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}}{\Gamma' \vdash \mathit{id}\, 3 : \mathtt{I}} \quad \dfrac{\dfrac{}{\Gamma' \vdash \mathit{id} : \mathtt{B} \to \mathtt{B}} \quad \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}}{\Gamma' \vdash \mathit{id}\, \mathtt{T} : \mathtt{B}}}{\Gamma, \{\mathit{id}{:}\forall\alpha.\alpha \to \alpha\} \vdash (\mathit{id}\, 3,\ \mathit{id}\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

$$\Gamma \vdash \mathtt{let}\ \mathit{id} = \lambda x.x\ \mathtt{in}\ (\mathit{id}\, 3,\ \mathit{id}\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}$$

Väga komplitseeritud tüübiga term

$$\begin{array}{l}
\mathtt{let}\ \mathit{pair} = \lambda xyz.z\, x\, y\ \mathtt{in} \\
\quad \mathtt{let}\ x_1 = \lambda y.\mathit{pair}\, y\, y\ \mathtt{in} \\
\quad\quad \mathtt{let}\ x_2 = \lambda y.x_1(x_1\, y)\ \mathtt{in} \\
\quad\quad\quad \mathtt{let}\ x_3 = \lambda y.x_2(x_2\, y)\ \mathtt{in} \\
\quad\quad\quad\quad \mathtt{let}\ x_4 = \lambda y.x_3(x_3\, y)\ \mathtt{in} \\
\quad\quad\quad\quad\quad \mathtt{let}\ x_5 = \lambda y.x_4(x_4\, y)\ \mathtt{in} \\
\quad\quad\quad\quad\quad\quad x_5(\lambda y.y)
\end{array}$$