Funktsionaalsete keelte semantika

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1 Programmeerimiskeel PCF

Vaatleme lihtsat funktsionaalset programmeerimiskeelt, mis on PCF-i variatsioon.

1.1 Süntaks

Süntaktilised kategooriad on $Expr$, $Pattern$, Var ja $Type$.
Avaldised:

$$Expr ::= Const | Var | \texttt{(}Expr\texttt{,}Expr\texttt{)} | \texttt{(}Expr\ \texttt{?}\ Expr\ \texttt{:}\ Expr\texttt{)} | (Expr\ Expr) | \texttt{(}\lambda Pattern\texttt{.}Expr\texttt{)} | \Upsilon Expr$$

Näidised:

$$Pattern ::= Var : Type \mid \texttt{(}Pattern\texttt{,}Pattern\texttt{)}$$

Muutujad:

$$Var$$

Tüübid:

$$Type ::= \texttt{int} \mid \texttt{bool} \mid \texttt{(}Type\texttt{->}Type\texttt{)} \mid \texttt{(}Type\texttt{*}Type\texttt{)}$$

Konstandid:

$$Const ::= Num \mid \texttt{+} \mid \texttt{-} \mid \texttt{*} \mid \texttt{/} \mid \texttt{=} \mid \texttt{<=} \mid \texttt{!}$$

Arvkonstandid (üks konstant iga mittenegatiivse täisarvu jaoks):

$$Num ::= \texttt{0} \mid \texttt{1} \mid \texttt{2} \mid \ldots$$

1.2 Staatiline semantika

Tüübikontekst $\Gamma \in$ Context on osaline funktsioon, mis seab mõnedele muutujatele vastavusse tüübi:

$$\text{Context} = Var \rightharpoonup Type$$

Muutuja staatiline semantika:

$$\mathcal{T}_V : Var \to (Type \to \text{Context} \to \text{Context}) \times (\text{Context} \rightharpoonup Type)$$

$$\mathcal{T}_V[\![x]\!] = (\mathcal{T}_{VN}[\![x]\!], \mathcal{T}_{VE}[\![x]\!])$$

$$\mathcal{T}_{VN} : Var \to Type \to \text{Context} \to \text{Context}$$

$$\mathcal{T}_{VN}[\![x]\!]\tau\ \Gamma = \Gamma[x \mapsto \tau]$$

$$\mathcal{T}_{VE} : Var \to \text{Context} \rightharpoonup Type$$

$$\mathcal{T}_{VE}[\![x]\!]\Gamma = \Gamma\ x$$

Tüübi staatiliseks semantikaks on see tüüp ise:

$$\mathcal{T}_{T} : Type \to Type$$

$$\mathcal{T}_{T}[\![\tau]\!] = \tau$$

Näidiste staatiline semantika:

$$\mathcal{T}_{N} : Pattern \to (\text{Context} \to \text{Context}) \times Type$$

$$\mathcal{T}_{N}[\![p]\!] = (\mathcal{T}_{NN}[\![p]\!], \mathcal{T}_{NT}[\![p]\!])$$

$$\mathcal{T}_{NN} : Pattern \to \text{Context} \to \text{Context}$$

$$\mathcal{T}_{NN}[\![x : \tau]\!] = \mathcal{T}_{VN}[\![x]\!]\tau$$

$$\mathcal{T}_{NN}[\![\texttt{(}p_1\texttt{,}p_2\texttt{)}]\!] = \mathcal{T}_{NN}[\![p_2]\!] \circ \mathcal{T}_{NN}[\![p_1]\!]$$

$$\mathcal{T}_{NT} : Pattern \to Type$$

$$\mathcal{T}_{NT}[\![x : \tau]\!] = \tau$$

$$\mathcal{T}_{NT}[\![\texttt{(}p_1\texttt{,}p_2\texttt{)}]\!] = \texttt{(}\mathcal{T}_{NT}[\![p_1]\!]\texttt{*}\mathcal{T}_{NT}[\![p_2]\!]\texttt{)}$$

Avaldiste staatiline semantika — tüübireeglid (kehtivad iga tüübikonteksti Γ korral ning iga $n \in Num; x \in Var; e, e_1, e_2 \in Expr; \tau, \tau_1, \tau_2 \in Type$ korral):

$$\frac{}{\Gamma \vdash x : \mathcal{T}_{VE}[\![x]\!]\Gamma}$$

$$\frac{}{\Gamma \vdash n : \texttt{int}}$$

$$\frac{}{\Gamma \vdash f : \texttt{(int->(int->int))}} \ (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}\})$$

$$\frac{}{\Gamma \vdash g : \texttt{(int->(int->bool))}} \ (g \in \{\texttt{=}, \texttt{<=}\})$$

$$\frac{}{\Gamma \vdash \texttt{!} : \texttt{(bool->bool)}}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash \texttt{(}e_1\texttt{,}e_2\texttt{)} : \texttt{(}\tau_1\texttt{*}\tau_2\texttt{)}}$$

$$\frac{\Gamma \vdash e_1 : \texttt{bool} \quad \Gamma \vdash e_2 : \tau \quad \Gamma \vdash e_3 : \tau}{\Gamma \vdash \texttt{(}e_1 \texttt{ ? } e_2 \texttt{ : } e_3\texttt{)} : \tau}$$

$$\frac{\Gamma \vdash e_1 : \texttt{(}\tau_1\texttt{->}\tau_2\texttt{)} \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash \texttt{(}e_1\ e_2\texttt{)} : \tau_2}$$

$$\frac{\mathcal{T}_{NN}[\![p]\!]\Gamma \vdash e : \tau}{\Gamma \vdash \texttt{(}\lambda p\texttt{.}e\texttt{)} : \texttt{(}\mathcal{T}_{NT}[\![p]\!]\texttt{->}\tau\texttt{)}}$$

$$\frac{\Gamma \vdash e : \texttt{(}\tau\texttt{->}\tau\texttt{)}}{\Gamma \vdash \Upsilon e : \tau}$$

Teoreem 1 *Igal avaldisel on ülimalt üks tüüp igas kontekstis.*

Tõestus. Induktsiooniga avaldise struktuuri järgi. Iga avaldise konstruktsiooni jaoks peale muutuja on täpselt üks tüübireegel, mis annab sellele avaldisele tüübi. Muutuja korral on reegel rakendatav siis, kui tüübikontekst annab antud muutujale tüübi. Seega on iga avaldise konstruktsiooni jaoks rakendatav ülimalt üks tüübireegel.

Vaatleme suvalist avaldist. Kui see avaldis on muutuja, siis annab tüübikontekst sellele ülimalt ühe tüübi, muul juhul on see konstrueeritud ühe konkreetse kontruktsiooniga, millele vastab ülimalt üks tüübireegel. Induktsiooni eelduse põhjal on selle avaldise igal alamavaldisel ülimalt üks tüüp. Seega kehtib reegli eeldus ülimalt ühe seal esinevate tüübimuutujate väärtustuse korral. Kuna iga reegli järelduses esinev muutuja esineb ka sama reegli eelduses (v.a muutuja reegli korral), siis on vaadeldava avaldise tüüp üheselt määratud.
$\square$

Teoreem 2 *Antud staatiline semantika on kompositsiooniline.*

Olgu $\mathcal{T} : Expr \to \text{Context} \rightharpoonup Type$ osaline funktsioon, mis seab igale avaldisele e vastavusse osalise funktsiooni $\mathcal{T}[\![e]\!]$ nii, et

$$\mathcal{T}[\![e]\!]\Gamma = \tau \Leftrightarrow \Gamma \vdash e : \tau$$

Tüüpidele annab semantika funktsioon $\mathcal{D}$, mis seab igale tüübile vastavusse tema väärtuste hulga:

$$\mathcal{D}[\![\texttt{int}]\!] = \mathbb{Z}_\perp$$

$$\mathcal{D}[\![\texttt{bool}]\!] = \mathbb{B}_\perp$$

$$\mathcal{D}[\![\tau_1\texttt{->}\tau_2]\!] = \{f : \mathcal{D}[\![\tau_1]\!] \to \mathcal{D}[\![\tau_2]\!] \mid f \text{ on pidev}\}$$
$$\mathcal{D}[\![\tau_1\texttt{*}\tau_2]\!] = \mathcal{D}[\![\tau_1]\!] \times \mathcal{D}[\![\tau_2]\!]$$

Siin $\mathbb{Z}_\perp = \mathbb{Z} \cup \{\perp_\mathbb{Z}\}$, $\mathbb{B}_\perp = \mathbb{B} \cup \{\perp_\mathbb{B}\}$ ja $\mathbb{B} = \{\mathbf{tt}, \mathbf{ff}\}$. Kõigi väärtuste hulk

$$\text{Value} = \bigcup_{\tau \in Type} \mathcal{D}[\![\tau]\!]$$

Seega $\mathcal{D} : Type \to \mathcal{P}(\text{Value})$.

Defineerime $\perp_\tau \in \mathcal{D}[\![\tau]\!]$ iga $\tau \in Type$ jaoks:

$$\perp_{\texttt{int}} = \perp_\mathbb{Z}$$
$$\perp_{\texttt{bool}} = \perp_\mathbb{B}$$
$$\perp_{\tau_1\texttt{*}\tau_2} = \texttt{(}\perp_{\tau_1}\texttt{,}\perp_{\tau_2}\texttt{)}$$
$$\perp_{\tau_1\texttt{->}\tau_2}\ x = \perp_{\tau_2}$$

Hulkadel $\mathcal{D}[\![\tau]\!]$, kus $\tau \in Type$ saab standardsel viisil defineerida järjestuse, mille suhtes need on täielikud järjestatud hulgad, ning $\perp_\tau$ on hulga $\mathcal{D}[\![\tau]\!]$ vähim element.

1.3 Näide

Näidisprogramm (faktoriaali arvutamine; siin on mõned sulud ära jäetud, lugedes funktsiooni rakendamise vasakassotsiatiivseks):

$$\Upsilon\texttt{(}\lambda\texttt{f:(int->int).(}\lambda\texttt{n:int.((= n 0) ? 1 : (* n (f (- n 1))))))}$$

Näitame, et sellel programmil (avaldisel) on olemas tüüp tühjas kontekstis.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\cdots}{\texttt{n} : \texttt{int} \vdash \texttt{(= n 0)} : \texttt{bool}} \quad \dfrac{}{\vdash \texttt{1} : \texttt{int}} \quad \dfrac{\cdots}{\texttt{f} : \texttt{(int->int)}, \texttt{n} : \texttt{int} \vdash \texttt{(* n (f (- n 1)))} : \texttt{int}}}{\texttt{f} : \texttt{(int->int)}, \texttt{n} : \texttt{int} \vdash \texttt{((= n 0) ? 1 : (* n (f (- n 1))))} : \texttt{int}}}{\texttt{f} : \texttt{(int->int)} \vdash \texttt{(}\lambda\texttt{n:int.((= n 0) ? 1 : (* n (f (- n 1)))))} : \texttt{(int->int)}}}{\vdash \texttt{(}\lambda\texttt{f:(int->int).(}\lambda\texttt{n:int.((= n 0) ? 1 : (* n (f (- n 1))))))} : \texttt{((int->int)->(int->int))}}}{\vdash \Upsilon\texttt{(}\lambda\texttt{f:(int->int).(}\lambda\texttt{n:int.((= n 0) ? 1 : (* n (f (- n 1))))))} : \texttt{(int->int)}}$$

1.4 Denotatsiooniline semantika

Keskkond $u \in \mathrm{Env}$ on osaline funktsioon, mis annab mõnedele muutujatele väärtuse:

$$\mathrm{Env} = Var \rightharpoonup \mathrm{Value}$$

Muutujate dünaamiline semantika:

$$\mathcal{S}_{VN} : Var \to \text{Value} \to \text{Env} \to \text{Env}$$

$$\mathcal{S}_{VN}[\![x]\!]a\ \epsilon = \epsilon[x \mapsto a]$$

$$\mathcal{S}_{VE} : Var \to \text{Env} \rightharpoonup \text{Value}$$

$$\mathcal{S}_{VE}[\![x]\!]\epsilon = \epsilon\ x$$

Näidiste dünaamiline semantika:

$$\mathcal{S}_N : Pattern \to \text{Value} \rightharpoonup \text{Env} \to \text{Env}$$

$$\mathcal{S}_N[\![p]\!] \in \mathcal{D}[\![\mathcal{T}_{NT}[\![p]\!]]\!] \to \text{Env} \to \text{Env}$$

$$\mathcal{S}_N[\![x : \tau]\!]a = \mathcal{S}_{VN}[\![x]\!]a$$

$$\mathcal{S}_N[\![(p_1, p_2)]\!](a_1, a_2) = \mathcal{S}_N[\![p_2]\!]a_2 \circ \mathcal{S}_N[\![p_1]\!]a_1$$

Avaldise semantika annab funktsioon

$$\mathcal{F} : Expr \to \text{Context} \times \text{Env} \rightharpoonup Type \times \text{Value},$$

$$\mathcal{F}[\![e]\!](\Gamma, \epsilon) = (\mathcal{T}[\![e]\!]\Gamma, \mathcal{S}[\![e]\!](\Gamma, \epsilon)),$$

kus

$$\mathcal{S} : \mathit{Expr} \to \text{Context} \times \text{Env} \rightharpoonup \text{Value}$$

Täisarvkonstandid:

$$\mathcal{S}[\![\mathsf{0}]\!]u = 0 \in \mathbb{Z}$$

$$\mathcal{S}[\![\mathsf{1}]\!]u = 1 \in \mathbb{Z}$$

$$\cdots$$

Ülejäänud konstandid:

$$\mathcal{S}[\![\mathsf{+}]\!]u, \mathcal{S}[\![\mathsf{-}]\!]u, \mathcal{S}[\![\mathsf{*}]\!]u, \mathcal{S}[\![\mathsf{/}]\!]u \in \mathbb{Z}_\perp \to \mathbb{Z}_\perp \to \mathbb{Z}_\perp$$

$$\mathcal{S}[\![\mathsf{+}]\!]u\ n_1\ n_2 = \begin{cases} n_1 + n_2, & \text{kui } n_1, n_2 \in \mathbb{Z} \\ \perp, & \text{kui } n_1 = \perp \text{ või } n_2 = \perp \end{cases}$$

$$\mathcal{S}[\![\mathsf{-}]\!]u\ n_1\ n_2 = \begin{cases} n_1 - n_2, & \text{kui } n_1, n_2 \in \mathbb{Z} \\ \perp, & \text{kui } n_1 = \perp \text{ või } n_2 = \perp \end{cases}$$

$$\mathcal{S}[\![\texttt{*}]\!]u\ n_1\ n_2 = \begin{cases} n_1 n_2, & \text{kui } n_1, n_2 \in \mathbb{Z} \\ \bot, & \text{kui } n_1 = \bot \text{ või } n_2 = \bot \end{cases}$$

$$\mathcal{S}[\![\texttt{/}]\!]u\ n_1\ n_2 = \begin{cases} \left\lfloor \frac{n_1}{n_2} \right\rfloor, & \text{kui } n_1, n_2 \in \mathbb{Z} \text{ ja } n_2 \neq 0 \\ \bot, & \text{kui } n_1 = \bot,\ n_2 = \bot \text{ või } n_2 = 0 \end{cases}$$

$$\mathcal{S}[\![\texttt{=}]\!]u, \mathcal{S}[\![\texttt{<=}]\!]u \in \mathbb{Z}_\bot \to \mathbb{Z}_\bot \to \mathbb{B}_\bot$$

$$\mathcal{S}[\![\texttt{=}]\!]u\ n_1\ n_2 = \begin{cases} \mathbf{tt}, & \text{kui } n_1, n_2 \in \mathbb{Z} \text{ ja } n_1 = n_2 \\ \mathbf{ff}, & \text{kui } n_1, n_2 \in \mathbb{Z} \text{ ja } n_1 \neq n_2 \\ \bot, & \text{kui } n_1 = \bot \text{ või } n_2 = \bot \end{cases}$$

$$\mathcal{S}[\![\texttt{<=}]\!]u\ n_1\ n_2 = \begin{cases} \mathbf{tt}, & \text{kui } n_1, n_2 \in \mathbb{Z} \text{ ja } n_1 \leq n_2 \\ \mathbf{ff}, & \text{kui } n_1, n_2 \in \mathbb{Z} \text{ ja } n_1 > n_2 \\ \bot, & \text{kui } n_1 = \bot \text{ või } n_2 = \bot \end{cases}$$

$$\mathcal{S}[\![\texttt{!}]\!]u \in \mathbb{B}_\bot \to \mathbb{B}_\bot$$

$$\mathcal{S}[\![\texttt{!}]\!]u\ b = \begin{cases} \mathbf{tt}, & \text{kui } b = \mathbf{ff} \\ \mathbf{ff}, & \text{kui } b = \mathbf{tt} \\ \bot, & \text{kui } b = \bot \end{cases}$$

Muutujad:

$$\mathcal{S}[\![x]\!](\Gamma,\epsilon) = \mathcal{S}_{VE}[\![x]\!]\epsilon$$

Paarid:

$$\mathcal{S}[\![\texttt{(}e_1\texttt{,}e_2\texttt{)}]\!]u = (\mathcal{S}[\![e_1]\!]u, \mathcal{S}[\![e_2]\!]u)$$

Tingimusavaldis:

$$\mathcal{S}[\![\texttt{(}b\ \texttt{?}\ e_1\ \texttt{:}\ e_2\texttt{)}]\!](\Gamma,\epsilon) = \begin{cases} \mathcal{S}[\![e_1]\!](\Gamma,\epsilon), & \text{kui } \mathcal{S}[\![b]\!]u = \mathbf{tt} \\ \mathcal{S}[\![e_2]\!](\Gamma,\epsilon), & \text{kui } \mathcal{S}[\![b]\!]u = \mathbf{ff} \\ \perp_{\mathcal{D}[\![\mathcal{T}[\![e_1]\!]\Gamma]\!]}, & \text{kui } \mathcal{S}[\![b]\!]u = \perp \end{cases}$$

Funktsiooni rakendamine:

$$\mathcal{S}[\![\texttt{(}e_1\ \ e_2\texttt{)}]\!]u = (\mathcal{S}[\![e_1]\!]u)(\mathcal{S}[\![e_2]\!]u)$$

Lambda-avaldis:

$$\mathcal{S}[\![\texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}]\!](\Gamma,\epsilon) \in \mathcal{D}[\![\mathcal{T}_{NT}[\![p]\!]]\!] \to \text{Value}$$

$$\mathcal{S}[\![\texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}]\!](\Gamma,\epsilon)\ a = \mathcal{S}[\![e]\!](\mathcal{T}_{NN}[\![p]\!]\Gamma, \mathcal{S}_N[\![p]\!]a\ \epsilon)$$

Püsipunktioperaator:

$$\mathcal{S}[\![\Upsilon e]\!](\Gamma, \epsilon) \in \mathcal{D}[\![\sigma]\!]$$

$$\mathcal{S}[\![\Upsilon e]\!](\Gamma, \epsilon) = \bigsqcup_{n=0}^{\infty} (\mathcal{S}[\![e]\!](\Gamma, \epsilon))^n(\bot_\sigma),$$

kus

$$\sigma = g(\mathcal{T}[\![e]\!]\Gamma)$$

$$g : Type \rightharpoonup Type$$

$$g(\tau\texttt{->}\tau) = \tau$$

Ütleme, et keskkond ϵ ühildub tüübikontekstiga Γ, kui

$$\mathrm{Arg}\ \epsilon = \mathrm{Arg}\ \Gamma$$

ning iga $x \in \mathrm{Arg}\ \epsilon$ korral

$$\epsilon\ x \in \mathcal{D}[\![\Gamma\ x]\!]$$

Tähistagu Env_Γ tüübikontekstiga Γ ühilduvate keskkondade hulka. Siis Env_Γ saab standardsel viisil muuta täielikuks järjestatud hulgaks.

Lemma 1 *Kehtivad järgmised ülemraja omadused:*

1. $\left(\bigsqcup_i f_i\right)(a) = \bigsqcup_i f_i(a)$
2. $\bigsqcup_i (a_i, b_i) = \left(\bigsqcup_i a_i, \bigsqcup_i b_i\right)$

Kehtivad järgmised pidevuse omadused:

1. *Kui* f *ja* g *on pidevad funktsioonid, siis funktsioon* $\lambda(a_1, a_2).(f\ a_1, g\ a_2)$ *on pidev.*
2. *Kui* f *on pidev ja iga* a *korral* $f(a)$ *on pidev, siis funktsioon* $\lambda(a, b).(f\ a\ b)$ *on pidev.*
3. *Kui* g *on pidev, siis funktsioon* $\lambda f.(g \circ f)$ *on pidev.*
4. *Kui* f *on pidev, siis funktsioon* $\lambda g.(g \circ f)$ *on pidev.*
5. *Funktsioon* $\lambda f.\lambda g.(g \circ f)$ *on pidev.*
6. *Funktsioon* $\lambda(f, g).(g \circ f)$ *on pidev.*
7. *Funktsioon* $\lambda f.(f\ a)$ *on pidev.*

8. *Funktsioon* $\lambda a.(a, a)$ *on pidev.*

9. *Funktsioon* $\lambda f.\lambda a.(f\ a)$ *on pidev.*

10. *Funktsioon* $\lambda(f, a).(f\ a)$ *on pidev.*

Lemma 2 *Iga* $x \in Var$, $\tau \in Type$, $a \in \mathcal{D}[\![\tau]\!]$, $\Gamma \in \mathrm{Context}$ *ja* $\epsilon \in \mathrm{Env}_\Gamma$ *korral*

- $\mathcal{S}_{VN}[\![x]\!]a\ \epsilon \in \mathrm{Env}_{\mathcal{T}_{VN}[\![x]\!]\tau\ \Gamma}$
- $\mathcal{S}_{VN}[\![x]\!]a \in \mathrm{Env}_\Gamma \to \mathrm{Env}_{\mathcal{T}_{VN}[\![x]\!]\tau\ \Gamma}$ *on pidev,*
- $\mathcal{S}_{VN}[\![x]\!] \in \mathcal{D}[\![\tau]\!] \to (\mathrm{Env}_\Gamma \to \mathrm{Env}_{\mathcal{T}_{VN}[\![x]\!]\tau\ \Gamma})$ *on pidev.*
- *Kui* $x \in \mathrm{Arg}\ \Gamma$, *siis* $\mathcal{S}_{VE}[\![x]\!] \in \mathrm{Env}_\Gamma \to \mathcal{D}[\![\mathcal{T}_{VE}[\![x]\!]\Gamma]\!]$ *on pidev funktsioon.*

Lemma 3 *Iga* $p \in Pattern$, $a \in \mathcal{D}[\![\mathcal{T}_{NT}[\![p]\!]]\!]$, $\Gamma \in \mathrm{Context}$ *ja* $\epsilon \in \mathrm{Env}_\Gamma$ *korral*

- $\mathcal{S}_N[\![p]\!]a\ \epsilon \in \mathrm{Env}_{\mathcal{T}_{NN}[\![p]\!]\Gamma}$,

- $\mathcal{S}_N[\![p]\!]a \in \mathrm{Env}_\Gamma \to \mathrm{Env}_{\mathcal{T}_{NN}[\![p]\!]\Gamma}$ *on pidev,*

- $\mathcal{S}_N[\![p]\!] \in \mathcal{D}[\![\mathcal{T}_{NT}[\![p]\!]]\!] \to (\mathrm{Env}_\Gamma \to \mathrm{Env}_{\mathcal{T}_{NN}[\![p]\!]\Gamma})$ *on pidev.*

Teoreem 3 *Kui $\mathcal{T}[\![e]\!]\Gamma$ on määratud, siis*

- *on iga kontekstiga Γ ühilduva keskkonna ϵ korral ka $\mathcal{S}[\![e]\!](\Gamma, \epsilon)$ (ja seega ka $\mathcal{F}[\![e]\!](\Gamma, \epsilon)$) määratud ning $\mathcal{S}[\![e]\!](\Gamma, \epsilon) \in \mathcal{D}[\![\mathcal{T}[\![e]\!]\Gamma]\!]$,*

- *funktsioon $\lambda\epsilon.\mathcal{S}[\![e]\!](\Gamma, \epsilon) \in \mathrm{Env}_\Gamma \to \mathcal{D}[\![\mathcal{T}[\![e]\!]\Gamma]\!]$ on pidev.*

TÕESTUS. Tõestame teoreemi väite induktsiooniga avaldise e struktuuri järgi.
$\square$

Järeldus 1 *Kui $\mathcal{T}[\![e]\!]\Gamma_0$ on määratud (avaldisel on tüüp tühjas kontekstis), siis ka $\mathcal{S}[\![e]\!](\Gamma_0, \epsilon_0)$ on määratud ning $\mathcal{S}[\![e]\!](\Gamma_0, \epsilon_0) \in \mathcal{D}[\![\mathcal{T}[\![e]\!]\Gamma_0]\!]$ (Γ_0 on tühi kontekst, ϵ_0 tühi keskkond).*

Seega igale avaldisele, millel on tüüp tühjas kontekstis, seatakse vastavusse väärtus, mis kuulub selle tüübi väärtuste hulka.

Teoreem 4 *Avaldiste semantika $\mathcal{F}$ on kompositsiooniline.*

TÕESTUS.

- Tüüpide semantika $\mathcal{D}$ on kompositsiooniline.
- Näidiste semantika on kompositsiooniline.
- Avaldise e tüüp $\mathcal{T}[\![e]\!]$ on defineeritud alamavaldiste ja -näidiste tüüpide kaudu (tüübireeglid on kompositsioonilised).
- Avaldise e dünaamiline semantika $\mathcal{S}[\![e]\!]$ on defineeritud avaldises esinevate näidiste semantika, alamavaldiste dünaamilise semantika ja alamavaldiste tüüpide kaudu.

$\square$

1.5 Struktuurne operatsiooniline semantika

Olgu $Var = Var_0 \cup Var_1$, $Var_0 \cap Var_1 = \emptyset$, kus Var_1 on lõpmatu hulk. Olgu $Expr'$ selliste avaldiste hulk, kus näidistes ei esine ühtegi muutujat hulgast Var_1 ja vabad muutujad on kõik hulgas Var_0 ning millele on lisatud $\mathbb{Z}$ ja $\mathbb{B}$ elemendid. Me anname struktuurse operatsioonilise semantika ainult hulgas $Expr'$ olevate avaldiste jaoks.

Olgu $\mathrm{Env}' = Var_1 \rightharpoonup Expr'$.

Olgu $\psi : \mathrm{Env}' \to Var_1$ selline funktsioon, et $\psi(\epsilon) \notin \mathrm{Arg}\ \epsilon$.

1.5.1 Näidiste semantika

Konfiguratsioonid on kujul $\langle p, e_1, \epsilon, e_2 \rangle$ või $\langle \epsilon, e_2 \rangle$, kus $p \in Pattern$, $e_1, e_2 \in Expr'$ ning $\epsilon \in \mathrm{Env}'$.

$$\langle x : \tau, e_1, \epsilon, e_2 \rangle \Rightarrow \langle \epsilon[\psi(\epsilon) \mapsto e_1], e_2[x/\psi(\epsilon)] \rangle$$

$$\frac{\langle p_2, e_2, \epsilon, e_3 \rangle \Rightarrow \langle p_2', e_2', \epsilon', e_3' \rangle}{\langle (p_1, p_2), (e_1, e_2), \epsilon, e_3 \rangle \Rightarrow \langle (p_1, p_2'), (e_1, e_2'), \epsilon', e_3' \rangle}$$

$$\frac{\langle p_2, e_2, \epsilon, e_3 \rangle \Rightarrow \langle \epsilon', e_3' \rangle}{\langle (p_1, p_2), (e_1, e_2), \epsilon, e_3 \rangle \Rightarrow \langle p_1, e_1, \epsilon', e_3' \rangle}$$

$$\frac{\langle e, \epsilon \rangle \Rightarrow \langle e', \epsilon' \rangle}{\langle (p_1, p_2), e, \epsilon, e_3 \rangle \Rightarrow \langle (p_1, p_2), e', \epsilon', e_3 \rangle} \; (e \notin \{\texttt{(}e_1\texttt{,}e_2\texttt{)} \mid e_1, e_2 \in \mathit{Expr}\})$$

1.5.2 Avaldiste semantika

Konfiguratsioonid on kujul $\langle e, \epsilon \rangle$ ning e, kus $e \in \mathit{Expr}'$ ja $\epsilon \in \mathrm{Env}'$.

$$\langle n, \epsilon \rangle \Rightarrow n \quad (n \in \mathbb{Z})$$

$$\langle b, \epsilon \rangle \Rightarrow b \quad (b \in \mathbb{B})$$

$$\langle \texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}, \epsilon \rangle \Rightarrow \texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}$$

$$\langle f, \epsilon \rangle \Rightarrow f \quad (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}, \texttt{!}\})$$

$$\langle \texttt{(}f\ \ e\texttt{)}, \epsilon \rangle \Rightarrow \texttt{(}f\ \ e\texttt{)} \quad (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}\})$$

$$\langle \texttt{0}, \epsilon \rangle \Rightarrow \langle 0, \epsilon \rangle$$

$$\langle \texttt{1}, \epsilon \rangle \Rightarrow \langle 1, \epsilon \rangle$$

$$\cdots$$

$$\langle \texttt{((+}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle n_1 + n_2, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z})$$

$$\langle \texttt{((-}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle n_1 - n_2, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z})$$

$$\langle \texttt{((*}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle n_1 n_2, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z})$$

$$\langle \texttt{((/}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \left\langle \left\lfloor \frac{n_1}{n_2} \right\rfloor, \epsilon \right\rangle \quad (n_1, n_2 \in \mathbb{Z}, n_2 \neq 0)$$

$$\langle \texttt{((=}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle \mathbf{tt}, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z}, n_1 = n_2)$$

$$\langle \texttt{((=}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle \mathbf{ff}, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z}, n_1 \neq n_2)$$

$$\langle \texttt{((<=}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle \mathbf{tt}, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z}, n_1 \leq n_2)$$

$$\langle \texttt{((<=}\ \ n_1\texttt{)}\ \ n_2\texttt{)}, \epsilon\rangle \Rightarrow \langle \mathbf{ff}, \epsilon\rangle \quad (n_1, n_2 \in \mathbb{Z}, n_1 > n_2)$$

$$\frac{\langle e_1, \epsilon\rangle \Rightarrow \langle e_1', \epsilon'\rangle}{\langle \texttt{((}f\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \Rightarrow \langle \texttt{((}f\ \ e_1'\texttt{)}\ \ e_2\texttt{)}, \epsilon'\rangle} \ (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}\})$$

$$\frac{\langle e_2, \epsilon \rangle \Rightarrow \langle e_2', \epsilon' \rangle}{\langle \texttt{((}f\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon \rangle \Rightarrow \langle \texttt{((}f\ \ e_1\texttt{)}\ \ e_2'\texttt{)}, \epsilon' \rangle}\ (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}\})$$

$$\frac{\langle e, \epsilon \rangle \Rightarrow \langle e', \epsilon' \rangle}{\langle \texttt{(!}\ \ e\texttt{)}, \epsilon \rangle \Rightarrow \langle \texttt{(!}\ \ e'\texttt{)}, \epsilon' \rangle}$$

$$\langle \texttt{(!}\ \ \mathbf{tt}\texttt{)}, \epsilon \rangle \Rightarrow \langle \mathbf{ff}, \epsilon \rangle$$

$$\langle \texttt{(!}\ \ \mathbf{ff}\texttt{)}, \epsilon \rangle \Rightarrow \langle \mathbf{tt}, \epsilon \rangle$$

$$\frac{\langle \epsilon\ x, \epsilon \rangle \Rightarrow \langle e', \epsilon' \rangle}{\langle x, \epsilon \rangle \Rightarrow \langle x, \epsilon'[x \mapsto e'] \rangle}$$

$$\frac{\langle \epsilon\ x, \epsilon \rangle \Rightarrow e'}{\langle x, \epsilon \rangle \Rightarrow \langle e', \epsilon \rangle}$$

$$\frac{\langle e_1, \epsilon \rangle \Rightarrow \langle e_1', \epsilon' \rangle}{\langle (e_1, e_2), \epsilon \rangle \Rightarrow \langle (e_1', e_2), \epsilon' \rangle}$$

$$\frac{\langle e_2, \epsilon \rangle \Rightarrow \langle e_2', \epsilon' \rangle}{\langle (e_1, e_2), \epsilon \rangle \Rightarrow \langle (e_1, e_2'), \epsilon' \rangle}$$

$$\frac{\langle b, \epsilon \rangle \Rightarrow \langle b', \epsilon' \rangle}{\langle (b \;?\; e_1 \;:\; e_2), \epsilon \rangle \Rightarrow \langle (b' \;?\; e_1 \;:\; e_2), \epsilon' \rangle}$$

$$\langle (\mathbf{tt} \;?\; e_1 \;:\; e_2), \epsilon \rangle \Rightarrow \langle e_1, \epsilon \rangle$$

$$\langle (\mathbf{ff} \;?\; e_1 \;:\; e_2), \epsilon \rangle \Rightarrow \langle e_2, \epsilon \rangle$$

$$\frac{\langle e_1, \epsilon \rangle \Rightarrow \langle e_1', \epsilon' \rangle}{\langle (e_1 \;\; e_2), \epsilon \rangle \Rightarrow \langle (e_1' \;\; e_2), \epsilon' \rangle}$$

$$\frac{\langle p, e_2, \epsilon, e_1 \rangle \Rightarrow \langle p', e_2', \epsilon', e_1' \rangle}{\langle ((\lambda p \,.\, e_1) \;\; e_2), \epsilon \rangle \Rightarrow \langle ((\lambda p' \,.\, e_1') \;\; e_2'), \epsilon' \rangle}$$

$$\frac{\langle p, e_2, \epsilon, e_1 \rangle \Rightarrow \langle \epsilon', e_1' \rangle}{\langle ((\lambda p \,.\, e_1) \;\; e_2), \epsilon \rangle \Rightarrow \langle e_1', \epsilon' \rangle}$$

$$\langle \Upsilon e, \epsilon \rangle \Rightarrow \langle (e \;\; \Upsilon e), \epsilon \rangle$$

1.5.3 Näited

Avaldis:

- `((`λ`f:int->int.((`λ`x:int.(f (f (f x)))) 3)) (`λ`x:int.((* x) x)))`

Keskkond:

-

Avaldis:

- `((`λ`x:int.(a0 (a0 (a0 x)))) 3)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`

Avaldis:

- `(a0 (a0 (a0 a1)))`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`

Avaldis:

- `((`λ`x:int.((* x) x)) (a0 (a0 a1)))`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `(a0 (a0 a1))`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((`λ`x:int.((* x) x)) (a0 a1))`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `(a0 a1)`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `((`λ`x:int.((* x) x)) a1)`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `((* a4) a4)`
- `a4` $\mapsto$ `a1`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `((* a4) a4)`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `((* 3) a4)`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `((* 3) 3)`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* a3) a3)`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* 9) a3)`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `((* 9) 9)`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* a2) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `81`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* 81) a2)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `81`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `((* 81) 81)`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `81`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

Avaldis:

- `6561`

Keskkond:

- `a0` $\mapsto$ `(`λ`x:int.((* x) x))`
- `a1` $\mapsto$ `3`
- `a2` $\mapsto$ `81`
- `a3` $\mapsto$ `9`
- `a4` $\mapsto$ `3`

1.6 Loomulik semantika

1.6.1 Näidiste semantika

Konfiguratsioonid on kujul $\langle p, e_1, \epsilon, e_2\rangle$ või $\langle \epsilon, e_2\rangle$, kus $p \in \mathit{Pattern}$, $e_1, e_2 \in \mathit{Expr}'$ ning $\epsilon \in \mathrm{Env}'$.

$$\langle x : \tau, e_1, \epsilon, e_2\rangle \rightarrow \langle \epsilon[\psi(\epsilon) \mapsto e_1], e_2[x/\psi(\epsilon)]\rangle$$

$$\frac{\langle p_2, e_2, \epsilon, e_3\rangle \rightarrow \langle \epsilon', e_3'\rangle \quad \langle p_1, e_1, \epsilon', e_3'\rangle \rightarrow \langle \epsilon'', e_3''\rangle}{\langle (p_1, p_2), (e_1, e_2), \epsilon, e_3\rangle \rightarrow \langle \epsilon'', e_3''\rangle}$$

1.6.2 Avaldiste semantika

Konfiguratsioonid on kujul $\langle e, \epsilon\rangle$, kus $e \in \mathit{Expr}'$ ja $\epsilon \in \mathrm{Env}'$.

$$\langle n, \epsilon\rangle \rightarrow n \quad (n \in \mathbb{Z})$$

$$\langle b, \epsilon\rangle \rightarrow b \quad (b \in \mathbb{B})$$

$$\langle \texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}, \epsilon\rangle \rightarrow \texttt{(}\lambda p\,\texttt{.}\,e\texttt{)}$$

$$\langle f, \epsilon\rangle \longrightarrow f \quad (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}, \texttt{!}\})$$

$$\langle \texttt{(}f\ \ e\texttt{)}, \epsilon\rangle \longrightarrow \texttt{(}f\ \ e\texttt{)} \quad (f \in \{\texttt{+}, \texttt{-}, \texttt{*}, \texttt{/}, \texttt{=}, \texttt{<=}\})$$

$$\langle \texttt{0}, \epsilon\rangle \longrightarrow 0$$

$$\langle \texttt{1}, \epsilon\rangle \longrightarrow 1$$

$$\ldots$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle \texttt{((+}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow n_1 + n_2}\ (n_1, n_2 \in \mathbb{Z})$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle \texttt{((-}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow n_1 - n_2}\ (n_1, n_2 \in \mathbb{Z})$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle \texttt{((*}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow n_1 n_2}\ (n_1, n_2 \in \mathbb{Z})$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle \texttt{((/}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow \left\lfloor \frac{n_1}{n_2} \right\rfloor}\ (n_1, n_2 \in \mathbb{Z}, n_2 \neq 0)$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle\texttt{((=}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{tt}}\ (n_1, n_2 \in \mathbb{Z}, n_1 = n_2)$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle\texttt{((=}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{ff}}\ (n_1, n_2 \in \mathbb{Z}, n_1 \neq n_2)$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle\texttt{((<=}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{tt}}\ (n_1, n_2 \in \mathbb{Z}, n_1 \leq n_2)$$

$$\frac{\langle e_1, \epsilon\rangle \longrightarrow n_1 \quad \langle e_2, \epsilon\rangle \longrightarrow n_2}{\langle\texttt{((<=}\ \ e_1\texttt{)}\ \ e_2\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{ff}}\ (n_1, n_2 \in \mathbb{Z}, n_1 > n_2)$$

$$\frac{\langle e, \epsilon\rangle \longrightarrow \mathbf{tt}}{\langle\texttt{(!}\ \ e\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{ff}}$$

$$\frac{\langle e, \epsilon\rangle \longrightarrow \mathbf{ff}}{\langle\texttt{(!}\ \ e\texttt{)}, \epsilon\rangle \longrightarrow \mathbf{tt}}$$

$$\frac{\langle \epsilon\ x, \epsilon\rangle \longrightarrow e}{\langle x, \epsilon\rangle \longrightarrow e}$$

$$\frac{\langle e_1, \epsilon\rangle \rightarrow e_1' \quad \langle e_2, \epsilon\rangle \rightarrow e_2'}{\langle (e_1, e_2), \epsilon\rangle \rightarrow (e_1', e_2')}$$

$$\frac{\langle b, \epsilon\rangle \rightarrow \mathbf{tt} \quad \langle e_1, \epsilon\rangle \rightarrow e_1'}{\langle (b \mathrel{?} e_1 \ : \ e_2), \epsilon\rangle \rightarrow e_1'}$$

$$\frac{\langle b, \epsilon\rangle \rightarrow \mathbf{ff} \quad \langle e_2, \epsilon\rangle \rightarrow e_2'}{\langle (b \mathrel{?} e_1 \ : \ e_2), \epsilon\rangle \rightarrow e_2'}$$

$$\frac{\langle e_1, \epsilon_1\rangle \rightarrow (\lambda p \,.\, e_3) \quad \langle p, e_2, \epsilon_1, e_3\rangle \rightarrow \langle \epsilon_2, e_3'\rangle \quad \langle e_3', \epsilon_2\rangle \rightarrow e_4}{\langle (e_1 \ \ e_2), \epsilon_1\rangle \rightarrow e_4}$$

$$\frac{\langle (e_1 \ \Upsilon e_1), \epsilon\rangle \rightarrow e_2}{\langle \Upsilon e_1, \epsilon\rangle \rightarrow e_2}$$

Teoreem 5 *Antud loomulik semantika on ühene.*

TÕESTUS. Induktsiooniga üle tuletuspuu struktuuri.
□

Seega indutseerib loomulik semantika semantilise funktsiooni

$$\mathcal{S}_{ns} : Expr' \to (Var_1 \rightharpoonup Expr') \rightharpoonup Expr'$$

$$\mathcal{S}_{ns}[\![e]\!]\epsilon = \begin{cases} e', & \text{kui } \langle e, \epsilon \rangle \to e' \\ \text{määramata} & \text{muul juhul} \end{cases}$$

Teoreem 6 *Kui $\mathcal{T}[\![e]\!]\Gamma$ on määratud, $\epsilon \in \mathrm{Env}' \cap \mathrm{Env}_\Gamma$, siis*

- $\mathcal{S}_{ns}[\![e]\!]\epsilon$ *on määratud*
- *kui $\mathcal{T}[\![e]\!]\Gamma \in \{\mathtt{int}, \mathtt{bool}\}$, siis $\mathcal{S}[\![e]\!](\Gamma, \epsilon) = \mathcal{S}_{ns}[\![e]\!]\epsilon$.*
- *$\mathcal{S}[\![e]\!](\Gamma, \epsilon) = \mathcal{S}_2[\![\mathcal{S}_{ns}[\![e]\!]\epsilon]\!](\Gamma, \epsilon)$, kus $\mathcal{S}_2$ on $\mathcal{S}$ laiendus hulgale $Expr'$ ($\mathbb{Z}$ ja $\mathbb{B}$ elementide korral identsus).*

TÕESTUS. Induktsiooniga üle avaldise struktuuri.
$\square$