

Exercises "MTAT.05.105 Type Theory"

Simply typed λ -calculus

1. In the following derivation, replace questionmarks with correct types and terms:

$$\frac{\frac{\frac{\frac{x:?, y:?, f:? \vdash ? : \beta}{x:?, y:?, f:? \vdash ? : \alpha} \quad \frac{x:?, y:?, f:? \vdash ? : \alpha \rightarrow \beta \rightarrow \gamma}{x:?, y:?, f:? \vdash ? : \beta \rightarrow \gamma}}{x:?, y:?, f:? \vdash ? : \gamma}}{x:?, y:?, f:? \vdash ? : (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma}}{x:? \vdash ? : \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma}}{\vdash ? : \alpha \rightarrow \beta \rightarrow (\alpha \rightarrow \beta \rightarrow \gamma) \rightarrow \gamma}$$

2. Find inhabitants (ie. *closed* terms) of the following types:

- (a) $((\alpha \rightarrow \beta \rightarrow \alpha) \rightarrow \alpha) \rightarrow \alpha$
 (b) $\beta \rightarrow ((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow \gamma$

3. Give a derivation of the following judgement:

$$\vdash \lambda x y. y (\lambda z. y x) : (\alpha \rightarrow \beta) \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta) \rightarrow \beta$$

4. Let $\mathbf{2} \equiv \lambda f z. f(f z)$ and $\mathbf{K} \equiv \lambda x y. x$. Show that the λ -term $\mathbf{2K}$ is not typable in Curry-style $\lambda \rightarrow$.

Second-order λ -calculus

5. Find a type for $\lambda x. x x x$ and give the corresponding Church-style λ -term.
 6. Show that all normal forms are typable in Curry-style $\lambda 2$.

The λ -cube

7. A raw environment Γ is called *valid* if it occurs in some derivable judgement. Show that an environment Γ is valid iff $\Gamma \vdash * : \square$.
 8. Let $\perp \equiv \Pi \alpha : *. \alpha$. Give derivation trees for the following judgements. What are the simplest systems of the λ -cube in which the judgements are valid?

$$\begin{array}{ll} \text{(a)} & \vdash (\lambda \beta : *. \lambda a : \perp. a \beta) : (\Pi \beta : *. \perp \rightarrow \beta) \\ \text{(b)} & A : *, P : A \rightarrow * \vdash (\lambda a : A. P a \rightarrow \perp) : (A \rightarrow *) \end{array}$$