The war on error

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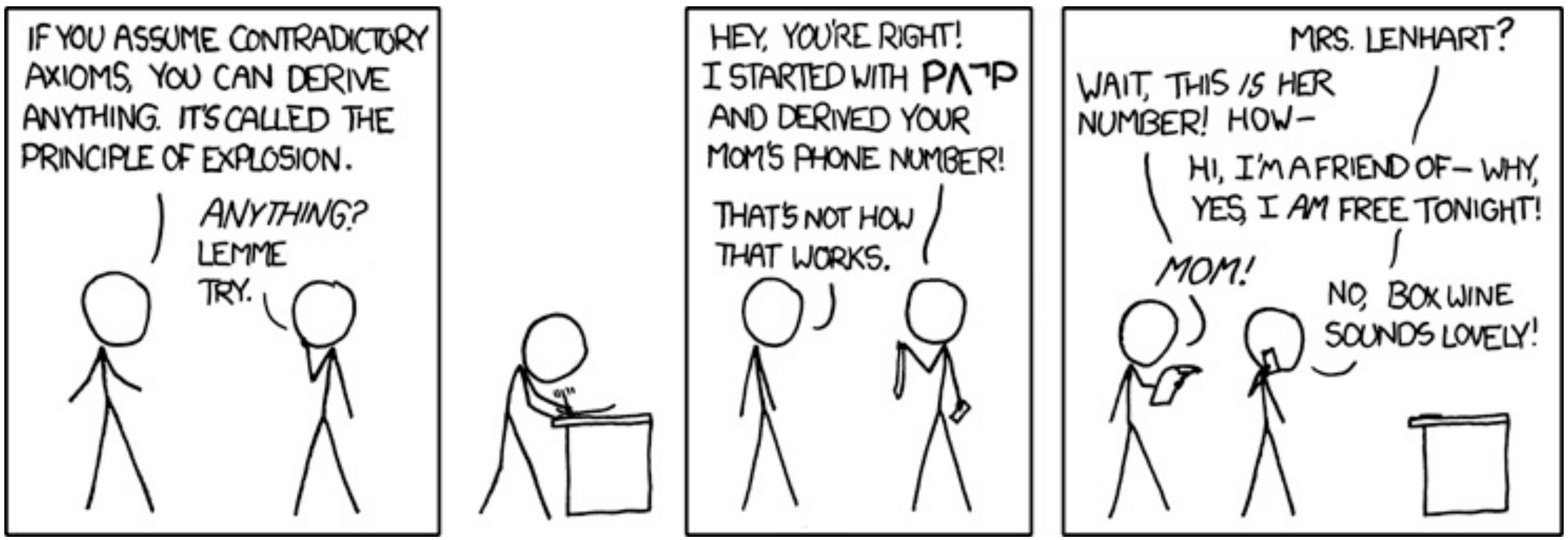
Principle of Explosion



Acknowledgements

- This weeks lectures and coursework borrow heavily from related courses by Thorsten Altenkich and Conor McBride

This lecture is about logic

- Don’t worry if you haven’t taken a logic course at all or have forgotten what you once knew.
- I will explain the style of logic (intuitionistic/constructive) used in Agda and other systems based on Martin-Löf’s type theory.

The last 100 years in the foundations of mathematics

- In 1900 there was the so-called ‘foundational crisis’. All of a sudden people became concerned that mathematics had no foundation.
- There were three competing schools:
  - Hilbert - wanted to justify mathematics by finitary means (prove consistency within the theories themselves)
  - Frege, Russell and Whitehead - reduce mathematics to logic.
  - Brouwer - thought the central principles of mathematics were flawed, they should be abandoned and mathematics reconstructed from the ground up by intuitionistic means.

What happened?

- Hilbert’s programme was shown to be in the pursuit of the impossible by Gödel’s incompleteness theorem.
- Russell and Whitehead’s work suffered from the same problem for the same reason.
- Brouwer made many enemies but his programme (Brouwer’s intuitionism) survives and was not vulnerable to Gödel’s theorem.

What is intuitionism?

- Mathematics is not concerned with statements which are a priori true of false.
- Mathematics concerns reasoning about mental constructions that we construct in our own minds.
- This is contrast to Platonism which suggests that there exists (in some intangible heavenly sense) all the theorems that are true (and their proofs) and all the theorems that are not true (and their counter examples).
- From an intuitionistic perspective mathematics only exists in our minds. The theorems and proofs we see on paper are not themselves mathematics.

Intuitionism and truth

- So, a theorem is not a priori true or false. Then how do we assert that it is true?
- We can show something is true by proving it.
- Intuitionism is therefore concerned with proof (what one can assert by intuition) and not with truth per say.
- But, this is all rather vague so let's restrict our attention to logic and then build up to a full-scale system for mathematics.

Brouwer's intuitionism

- Start again with a new mathematics
- The consequences of the intuitionistic viewpoint are that we must reject some classical laws:
  - Excluded middle: $A \vee \neg A$
  - Proof by contradiction: $A \Leftrightarrow \neg\neg A$
- He did not work formally, he hated formalism and formalists. He wouldn't like Agda much I don't think...

BHK Interpretation

Brouwer, Heyting and Kolmogorov

- A proof of A → B is a **function** which converts proofs of A into proofs of B
- A proof of A ∧ B is a **pair** of a proof of A and a proof of B
- A proof of A ∨ B is **either** a proof of A or it is a proof of B
- A proof of ∃x. P x means a pair of a **witness** x and a proof that x satisfies the predicate P
- A proof of ∀x. P x is a function which converts a proof x into a proof that x satisfies the predicate P

Heyting
Intuitionistic
logic

BHK continued

- The proposition True has a **trivial proof**.
- The proposition False has **no proof**.
- If we have a proof of False we can derive **any other propostion**.
- negation ¬A is defined as A → False. Having a proof of A leads to a **contradiction**.

Implication viewed intuitionisticly

- A proof of a A → B is the same a program that takes an A as input and produces a B as output.
- Why?
  - Because underlying both proofs and programs are the idea of an algorithm/ procedure that we can follow intuitively.

What have we gained? What have we lost?

- We have gained an intuitive understanding of logic which is very natural for computer scientists - proofs are just programs.
- We have lost some equations of classical logic
  - Excluded middle
  - proof by contradiction
  - 3rd de Morgan law
- But! All is not lost. For any classical proof A, it's double negation $\neg\neg A$ holds intuitionistically.

Intuitionistic logic

Proof	=	Program
Proposition	=	Specification
Proposition	≠	Type

because we predicates can only talk about propositions

Is intuitionistic logic enough?

We can prove theorems by writing programs:

```
d : A ∧ (B ∨ C) → (A ∧ B) ∨ (A ∧ C)
d (a , left  b) = left  (a , b)
d (a , right c) = right (a , c)
```

But we cannot reason about proofs:

```
e  :  ∀ p , q : A ∧ (B ∨ C) .
                (d p == d q) → (p == q)
```

a relation (a binary predicate) not a proposition

A full-scale intuitionistic system

- Can quantify over proofs of a proposition (Sigma and Pi types)
- Satisfies BHK interpretation of logic
- Started by Howard
  - “The formulae-as-types notion of construction”
- Finished by Martin-Löf
  - “an intuitionistic theory of types”

A full-scale intuitionistic system (2)

Proof	=	Program
Proposition	=	Type
Cut elimination	=	Normalisation
Proof checking	=	Type checking

The whole picture is the so-called Curry-Howard correspondence. Tait also deserves some credit.

Computerized mental mathematics?

- If a proof exists only in our minds then what happens when we run it on a computer?
- I would say that we can implement on a computer the intuitive steps that we carry out in our minds when we construct a proof. When the computer ‘runs’ the proof it produces a ‘proof object’ (an expression in normal form which can be independently checked) and this is the analogue of a mental construction.

Live coding

- In the last coursework we defined Boolean logic as a datatype in Agda and then proved some properties about this definition.
- Now we will use Agda's logic directly...