MTAT.05.105 Type Theory

Curry-Howard correspondence

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Curry-Howard correspondence (*en.wikipedia.org*)

The Curry–Howard correspondence is the direct relationship between computer programs and proofs in constructive mathematics. Also known as Curry–Howard isomorphism, proofs-as-programs correspondence and formulae-as-types correspondence, it refers to the generalization of a syntactic analogy between systems of formal logic and computational calculi that was first discovered by the American mathematician Haskell Curry and logician William Alvin Howard.

Classical vs. constructive logic

Classical logic

- Every proposition is either true or false.
- Concerned with:

"Whether a given proposition is true or not?"

Constructive logic

- Proposition is true only if we can prove it.
- Concerned with:

"How a given proposition becomes true?"

Classical tautologies not provable constructively

$A \vee \neg A \qquad \neg\neg A \supset A \qquad ((A \supset B) \supset A) \supset A$

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Natural deduction

- General form of inference rules:

$$\frac{P_1 \quad P_2 \quad \ldots \quad P_n}{P_0}$$

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  - $P_1, \ldots, P_n$ are premises, P_0 is a conclusion.
  - If $n = 0$ (no premise), the inference rule is an axiom.

Natural deduction

- General form of inference rules:

$$\frac{P_1 \quad P_2 \quad \ldots \quad P_n}{P_0}$$

- For each connective ($\wedge$, $\vee$, ...) two kinds of rules.
- Introduction rules:
  - Connective appears in the conclusion P_0.
  - *"How to estabish a proof?"*
- Elimination rules:
  - Connective appears in a premis P_i.
  - *"How to exploit an existing proof?"*
- Note: Usually a connective has a single introduction and a single elimination rule, but some connectives may have several rules of the same kind or no rules of certain kind.

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

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- Inference rules for implication:
  - Introduction:

$$\dfrac{\begin{array}{c}\overline{P_1}^{\,x} \\ \vdots \\ P_2\end{array}}{P_1 \supset P_2}\ \supset\mathsf{I}^{x}$$

  - Elimination:

$$\dfrac{P_1 \supset P_2 \qquad P_1}{P_2}\ \supset\mathsf{E}$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for conjunction:
  - Introduction:

$$\frac{P_1 \quad P_2}{P_1 \wedge P_2}\ \wedge\mathsf{I}$$

  - Elimination:

$$\frac{P_1 \wedge P_2}{P_1}\ \wedge\mathsf{E}_L \qquad\qquad \frac{P_1 \wedge P_2}{P_2}\ \wedge\mathsf{E}_R$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for disjunction:
  - Introduction:

$$\frac{P_1}{P_1 \vee P_2}\ \vee\mathsf{I}_L \qquad\qquad \frac{P_2}{P_1 \vee P_2}\ \vee\mathsf{I}_R$$

  - Elimination:

$$\frac{P_1 \vee P_2 \quad \begin{array}{c}\overline{P_1}^{\,x} \\ \vdots \\ P_0\end{array} \quad \begin{array}{c}\overline{P_2}^{\,y} \\ \vdots \\ P_0\end{array}}{P_0}\ \vee\mathsf{E}^{x,y}$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for truth and falsehood:
  - Introduction:

$$\frac{}{\top}\ \top\mathsf{I}$$

  - Elimination:

$$\frac{\bot}{P}\ \bot\mathsf{E}$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for truth and falsehood:
  - Introduction:

$$\frac{}{\top}\ \top\mathsf{I}$$

  - Elimination:

$$\frac{\bot}{P}\ \bot\mathsf{E}$$

  - Truth can be defined as a *"syntactic sugar"*:

$$\top \quad \equiv \quad \bot \supset \bot$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for negation:
  - Introduction:

$$\dfrac{\begin{array}{c}\overline{P}^{\,x} \\ \vdots \\ \bot\end{array}}{\neg P}\ \neg I^{x}$$

  - Elimination:

$$\dfrac{\neg P \quad P}{\bot}\ \neg E$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Inference rules for negation:
  - Introduction:

$$\dfrac{\begin{array}{c}\overline{P}^{\,x} \\ \vdots \\ \bot\end{array}}{\neg P}\ \neg\mathsf{I}^{x}$$

  - Elimination:

$$\dfrac{\neg P \qquad P}{\bot}\ \neg\mathsf{E}$$

  - Negation can be defined as a *"syntactic sugar"*:

$$\neg P \quad \equiv \quad P \supset \bot$$

Propositional logic

- Syntax:

$$P ::= A \mid P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P$$

- Presented rules give intuitionistic propositional logic IPC.
- Sometimes we may use smaller fragments:
  - System without the $\bot\mathrm{E}$ rule is minimal propositional logic $ND(\supset, \wedge, \vee)$.
  - System with only implication rules (implicational fragment) $ND(\supset)$.
- Classical propositional logic can be obtained by adding the double negation elimination rule:

$$\frac{\neg\neg P}{P}$$

Propositional logic

Example proof (1):

$$A \land B \supset B \land A$$

Propositional logic

Example proof (1):

$$\dfrac{\dfrac{\overline{A \wedge B}^{\,x}}{\vdots}\qquad B \wedge A}{A \wedge B \supset B \wedge A}\ \supset\mathsf{I}^{x}$$

Propositional logic

Example proof (1):

$$\dfrac{\dfrac{\begin{array}{c}\overline{A \wedge B}^{\;x}\\ \vdots\\ B\end{array}\qquad\qquad\begin{array}{c}\overline{A \wedge B}^{\;x}\\ \vdots\\ A\end{array}}{B \wedge A}\;\wedge\mathsf{I}}{A \wedge B \supset B \wedge A}\;\supset\mathsf{I}^{x}$$

Propositional logic

Example proof (1):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge B}\,x}{B}\,\wedge\mathsf{E}_R \qquad \dfrac{\dfrac{}{A \wedge B}\,x}{A}\,\wedge\mathsf{E}_L}{B \wedge A}\,\wedge\mathsf{I}}{A \wedge B \supset B \wedge A}\,\supset\mathsf{I}^x$$

Propositional logic

Example proof (2):

$$(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)$$

Propositional logic

Example proof (2):

$$\dfrac{}{(A \supset B) \wedge (A \supset C)}\ x$$

$$\vdots$$

$$\dfrac{A \supset (B \wedge C)}{(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)} \supset\mathsf{I}^x$$

Propositional logic

Example proof (2):

$$\dfrac{\dfrac{\begin{array}{c} \dfrac{}{A}\,{}^{y} \qquad \dfrac{}{(A \supset B) \wedge (A \supset C)}\,{}^{x} \\ \vdots \\ B \wedge C \end{array}}{A \supset (B \wedge C)}\ {\supset}\mathsf{I}^{y}}{(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)}\ {\supset}\mathsf{I}^{x}$$

Propositional logic

Example proof (2):

$$
\cfrac{\cfrac{\cfrac{
\begin{array}{cc}
\cfrac{}{A}\,y & \cfrac{}{(A \supset B) \wedge (A \supset C)}\,x \\
\vdots & \vdots \\
\multicolumn{2}{c}{B}
\end{array}
\qquad
\begin{array}{cc}
\cfrac{}{A}\,y & \cfrac{}{(A \supset B) \wedge (A \supset C)}\,x \\
\vdots & \vdots \\
\multicolumn{2}{c}{C}
\end{array}
}{B \wedge C}\,\wedge\mathsf{I}}{A \supset (B \wedge C)}\,\supset\mathsf{I}^{y}}{(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)}\,\supset\mathsf{I}^{x}
$$

Propositional logic

Example proof (2):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\overline{A}^{\,y} \qquad \begin{array}{c}\overline{(A \supset B) \wedge (A \supset C)}^{\,x} \\ \vdots \\ A \supset B\end{array}}{B}\,{\supset}\mathsf{E} \qquad \dfrac{\overline{A}^{\,y} \qquad \begin{array}{c}\overline{(A \supset B) \wedge (A \supset C)}^{\,x} \\ \vdots \\ A \supset C\end{array}}{C}\,{\supset}\mathsf{E}}{B \wedge C}\,{\wedge}\mathsf{I}}{A \supset (B \wedge C)}\,{\supset}\mathsf{I}^{y}}{(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)}\,{\supset}\mathsf{I}^{x}
$$

Propositional logic

Example proof (2):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A}\,y \qquad \dfrac{\dfrac{}{(A \supset B) \wedge (A \supset C)}\,x}{A \supset B}\,\wedge\mathsf{E}_L}{B}\,\supset\mathsf{E} \qquad \dfrac{\dfrac{}{A}\,y \qquad \dfrac{\dfrac{}{(A \supset B) \wedge (A \supset C)}\,x}{A \supset C}\,\wedge\mathsf{E}_R}{C}\,\supset\mathsf{E}}{B \wedge C}\,\wedge\mathsf{I}}{A \supset (B \wedge C)}\,\supset\mathsf{I}^y}{(A \supset B) \wedge (A \supset C) \supset A \supset (B \wedge C)}\,\supset\mathsf{I}^x$$

Propositional logic

Example proof (3):

$$A \supset B \supset B$$

Propositional logic

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$$\frac{B \supset B \qquad (B \supset B) \supset A \supset B \supset B}{A \supset B \supset B} \supset E$$

Propositional logic

Example proof (3):

$$\dfrac{\dfrac{\dfrac{}{B}\,x \quad \vdots \quad B}{B \supset B}\supset\mathsf{I}^x \qquad (B \supset B) \supset A \supset B \supset B}{A \supset B \supset B}\supset\mathsf{E}$$

Propositional logic

Example proof (3):

$$\dfrac{\dfrac{\dfrac{}{B}\,x}{B \supset B}\,{\supset}\mathsf{I}^{x} \qquad (B \supset B) \supset A \supset B \supset B}{A \supset B \supset B}\,{\supset}\mathsf{E}$$

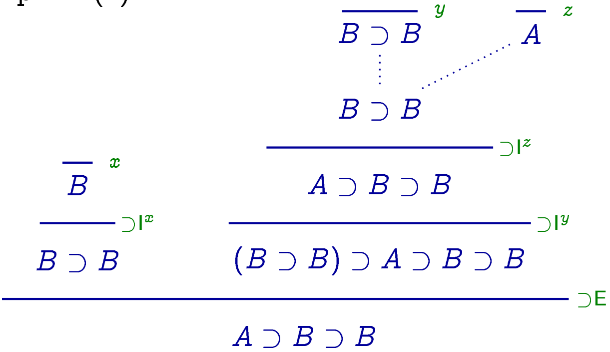
Propositional logic

Example proof (3):

$$
\dfrac{\dfrac{\overline{B}^{\,x}}{B \supset B}\supset\mathsf{I}^x \qquad \dfrac{\begin{array}{c}\overline{B \supset B}^{\,y} \\ \vdots \\ A \supset B \supset B\end{array}}{(B \supset B) \supset A \supset B \supset B}\supset\mathsf{I}^y}{A \supset B \supset B}\supset\mathsf{E}
$$

Propositional logic

Example proof (3):



Propositional logic

Example proof (3):

$$\dfrac{\dfrac{\overline{B}^{\,x}}{B \supset B}\,{\supset}\mathsf{I}^x \qquad \dfrac{\dfrac{\overline{B \supset B}^{\,y}}{A \supset B \supset B}\,{\supset}\mathsf{I}^z}{(B \supset B) \supset A \supset B \supset B}\,{\supset}\mathsf{I}^y}{A \supset B \supset B}\,{\supset}\mathsf{E}$$

Propositional logic

Example (3) — alternative proof:

$$A \supset B \supset B$$

Propositional logic

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$$\dfrac{\dfrac{\overline{A}^{\,x}}{\vdots}\qquad}{B \supset B}$$

$$\dfrac{B \supset B}{A \supset B \supset B}\ \supset\mathsf{I}^{x}$$

Propositional logic

Example (3) — alternative proof:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A}\,^{x} \qquad\qquad \overline{B}\,^{y} \\ \ddots \qquad \unicode{x22F0} \\ B\end{array}}{B \supset B}\,{\supset}\mathsf{I}^{y}}{A \supset B \supset B}\,{\supset}\mathsf{I}^{x}$$

Propositional logic

Example (3) — alternative proof:

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Proof normalization

- Theorem: Every provable proposition has a normal proof.

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- Normalization rules — implication:

$$\dfrac{\begin{array}{c}\vdots\ \Sigma\\ S\end{array} \qquad \dfrac{\begin{array}{c}\overline{S}\\ \vdots\ \Pi\\ P\end{array}}{S \supset P}}{P} \quad \to \quad \begin{array}{c}\vdots\ \Sigma\\ S\\ \vdots\ \Pi\\ P\end{array}$$

Proof normalization

- Theorem: Every provable proposition has a normal proof.
- Normalization rules — conjunction:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Sigma \\ P_1\end{array} \qquad \begin{array}{c}\vdots\ \Pi \\ P_2\end{array}}{P_1 \wedge P_2}}{P_1} \quad \rightarrow \quad \begin{array}{c}\vdots\ \Sigma \\ P_1\end{array}$$

Proof normalization

- Theorem: Every provable proposition has a normal proof.
- Normalization rules — disjunction:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \Theta\\ P_1\end{array}}{P_1 \vee P_2} \qquad \begin{array}{c}\overline{P_1}\\ \vdots\ \Sigma \\ S\end{array} \qquad \begin{array}{c}\overline{P_2}\\ \vdots\ \Pi\\ S\end{array}}{S} \qquad \to \qquad \begin{array}{c}\vdots\ \Theta\\ P_1\\ \vdots\ \Sigma\\ S\end{array}$$

Curry-Howard isomorphism

- Theorem:

(i) If $\Gamma \vdash M : \varphi$ in $\lambda(\to, \times, +)$, then $|\Gamma| \vdash \varphi$ in $ND(\supset, \wedge, \vee)$, where $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$.

(ii) If $\Gamma \vdash \varphi$ in $ND(\supset, \wedge, \vee)$, then there exists a term M in $\lambda(\to, \times, +)$, s.t. $\Delta \vdash M : \varphi$, where $\Delta = \{x_\varphi : \varphi \mid \varphi \in \Gamma\}$.

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- Curry-Howard correspondence:

Proposition	Type
$\bot$	Void
$\top$	Unit
$A \supset B$	$A \rightarrow B$
$A \wedge B$	$A \times B$
$A \vee B$	$A + B$

Curry-Howard isomorphism

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(i) If $\Gamma \vdash M : \varphi$ in $\lambda(\rightarrow, \times, +)$, then $|\Gamma| \vdash \varphi$ in $ND(\supset, \wedge, \vee)$, where $|\Gamma| = \{\varphi \mid (x : \varphi) \in \Gamma\}$.

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- Curry-Howard correspondence:

Intuitionistic logic	Typed λ-calculus
Proposition	Type
Propositional variable	Type variable
Proof	Term
Hypothesis	Term variable
Logical connective	Type constructor
Provability	Type inhabitation
Proof normalization	Reduction

First-order predicate logic

- Syntax:

$$
\begin{array}{llll}
t & ::= & v \mid f(t, \ldots, t) & \text{terms (atomic formulas)} \\
P & ::= & Q(t, \ldots, t) & \text{atomic predicates} \\
 & \mid & P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P & \\
 & \mid & \forall v.P \mid \exists v.P & \text{1st-order quantifiers}
\end{array}
$$

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\end{array}
$$

- Inference rules for universal quantification:

$$
\frac{P[v \mapsto w]}{\forall v.P}\ \forall^1\mathsf{I}^{\star}
\qquad\qquad
\frac{\forall v.P}{P[v \mapsto t]}\ \forall^1\mathsf{E}
$$

$\star$ variable w in $\forall^1\mathsf{I}$ must be fresh!

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\end{array}
$$

- Inference rules for existential quantification:

$$
\frac{P[v \mapsto t]}{\exists v.P}\ \exists^1\mathsf{I}
\qquad\qquad
\frac{\exists v.P_1 \qquad \begin{array}{c}\overline{P_1[v \mapsto w]}^{\,x} \\ \vdots \\ P_0\end{array}}{P_0}\ \exists^1\mathsf{E}^{x\star}
$$

$\star$ variable w in $\exists^1\mathsf{E}^x$ must be fresh!

First-order predicate logic

Example proof (1):

$$(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)$$

First-order predicate logic

Example proof (1):

$$\dfrac{}{\exists u.\forall v.Q(u,v)}\;x$$

$$\vdots$$

$$\dfrac{\forall v.\exists u.Q(u,v)}{(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)}\;\supset\mathsf{I}^{x}$$

First-order predicate logic

Example proof (1):

$$\dfrac{\dfrac{\dfrac{}{\exists u.\forall v.Q(u,v)}\,x \quad \cdots \quad \exists u.Q(u,v')}{\forall v.\exists u.Q(u,v)}\,\forall^1 \mathsf{I}}{(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)}\,\supset\mathsf{I}^x$$

First-order predicate logic

Example proof (1):

$$
\dfrac{
  \dfrac{
    \dfrac{}{\exists u.\forall v.Q(u,v)}\,x
    \qquad
    \begin{array}{c}
      \dfrac{}{\forall v.Q(u',v)}\,y \\
      \vdots \\
      \exists u.Q(u,v')
    \end{array}
  }{\exists u.Q(u,v')}\,\exists^1\mathsf{E}^y
}{
  \dfrac{\forall v.\exists u.Q(u,v)}{(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)}\,\supset\mathsf{I}^x
}\,\forall^1\mathsf{I}
$$

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$$
\dfrac{
  \dfrac{}{\exists u.\forall v.Q(u,v)}\,x
  \qquad
  \dfrac{
    \begin{array}{c}
      \dfrac{}{\forall v.Q(u',v)}\,y \\
      \vdots \\
      Q(u',v')
    \end{array}
  }{\exists u.Q(u,v')}\,\exists^1\mathsf{I}
}{
  \dfrac{
    \dfrac{\exists u.Q(u,v')}{\forall v.\exists u.Q(u,v)}\,\forall^1\mathsf{I}
  }{(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)}\,\supset\mathsf{I}^x
}\,\exists^1\mathsf{E}^y
$$

First-order predicate logic

Example proof (1):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\exists u.\forall v.Q(u,v)}\,x \qquad \dfrac{\dfrac{\dfrac{}{\forall v.Q(u',v)}\,y}{Q(u',v')}\,\forall^1\mathsf{E}}{\exists u.Q(u,v')}\,\exists^1\mathsf{I}}{\exists u.Q(u,v')}\,\exists^1\mathsf{E}^y}{\forall v.\exists u.Q(u,v)}\,\forall^1\mathsf{I}}{(\exists u.\forall v.Q(u,v)) \supset \forall v.\exists u.Q(u,v)}\,\supset\mathsf{I}^x$$

First-order predicate logic

Example proof (2):

$$(\exists v.Q(v)) \supset \neg\forall v.\neg Q(v)$$

First-order predicate logic

Example proof (2):

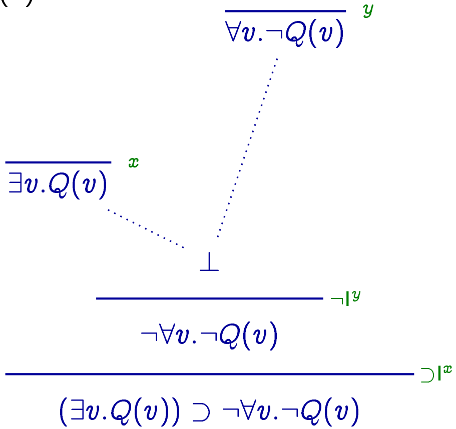
$$\dfrac{}{\exists v.Q(v)}\;x$$

$$\vdots$$

$$\dfrac{\neg\forall v.\neg Q(v)}{(\exists v.Q(v)) \supset \neg\forall v.\neg Q(v)}\;\supset\mathsf{I}^{x}$$

First-order predicate logic

Example proof (2):



First-order predicate logic

Example proof (2):

$$
\dfrac{
  \dfrac{
    \dfrac{}{\exists v.Q(v)}\,x
    \qquad
    \dfrac{
      \overline{\forall v.\neg Q(v)}^{\,y}
      \quad
      \overline{Q(v')}^{\,z}
    }{\bot}\,\vdots
  }{\bot}\,\exists^1\mathsf{E}^z
}{
  \dfrac{}{\neg\forall v.\neg Q(v)}
}\,\neg\mathsf{I}^y
$$

$$
\dfrac{\neg\forall v.\neg Q(v)}{(\exists v.Q(v)) \supset \neg\forall v.\neg Q(v)}\,\supset\!\mathsf{I}^x
$$

First-order predicate logic

Example proof (2):

$$
\dfrac{
  \dfrac{\dfrac{}{\exists v.Q(v)}\,x
  \qquad
  \dfrac{
    \begin{array}{c}\dfrac{}{\forall v.\neg Q(v)}\,y \\ \vdots \\ \neg Q(v')\end{array}
    \qquad
    \dfrac{}{Q(v')}\,z
  }{\bot}\,\neg\mathsf{E}}{\bot}\,\exists^1\mathsf{E}^z
}{\neg\forall v.\neg Q(v)}\,\neg\mathsf{I}^y
$$

$$
\dfrac{\neg\forall v.\neg Q(v)}{(\exists v.Q(v)) \supset \neg\forall v.\neg Q(v)}\,\supset\mathsf{I}^x
$$

First-order predicate logic

Example proof (2):

$$
\dfrac{\dfrac{\dfrac{\overline{\exists v.Q(v)}^{\,x} \qquad \dfrac{\dfrac{\overline{\forall v.\neg Q(v)}^{\,y}}{\neg Q(v')}\,\forall^1\mathsf{E} \qquad \overline{Q(v')}^{\,z}}{\bot}\,\neg\mathsf{E}}{\bot}\,\exists^1\mathsf{E}^z}{\neg\forall v.\neg Q(v)}\,\neg\mathsf{I}^y}{(\exists v.Q(v)) \supset \neg\forall v.\neg Q(v)}\,\supset\mathsf{I}^x
$$

Second-order predicate logic

- Syntax:

$$
\begin{array}{rcll}
P & ::= & Q(t, \ldots, t) & \text{atomic predicates} \\
& | & X(t, \ldots, t) & \text{propositional variables} \\
& | & P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P & \\
& | & \forall v.P \mid \exists v.P & \text{1st-order quantifiers} \\
& | & \forall X.P \mid \exists X.P & \text{2nd-order quantifiers}
\end{array}
$$

Second-order predicate logic

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$$\begin{array}{rcll} P & ::= & Q(t, \ldots, t) & \text{atomic predicates} \\ & | & X(t, \ldots, t) & \text{propositional variables} \\ & | & P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P & \\ & | & \forall v.P \mid \exists v.P & \text{1st-order quantifiers} \\ & | & \forall X.P \mid \exists X.P & \text{2nd-order quantifiers} \end{array}$$

- Inference rules for universal quantification:

$$\frac{P[X \mapsto Y]}{\forall X.P}\ \forall^2 I^{\star} \qquad \frac{\forall X.P_0}{P_0[X \mapsto P_1]}\ \forall^2 E$$

$\star$ variable Y in $\forall^2 I$ must be fresh!

Second-order predicate logic

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$$
\begin{array}{rcll}
P & ::= & Q(t,\ldots,t) & \text{atomic predicates} \\
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 & | & \forall v.P \mid \exists v.P & \text{1st-order quantifiers} \\
 & | & \forall X.P \mid \exists X.P & \text{2nd-order quantifiers}
\end{array}
$$

- Inference rules for existential quantification:

$$
\frac{P_0[X \mapsto P_1]}{\exists X.P_0}\ \exists^2\mathsf{I}
\qquad\qquad
\frac{\exists X.P_1 \qquad \begin{array}{c}\overline{P_1[X \mapsto Y]}^{\,x} \\ \vdots \\ P_0\end{array}}{P_0}\ \exists^2\mathsf{E}^{x\star}
$$

$\star$ variable Y in $\exists^2\mathsf{E}^x$ must be fresh!

Second-order predicate logic

- Syntax:

$$
\begin{array}{rcll}
P & ::= & Q(t,\ldots,t) & \text{atomic predicates} \\
 & \mid & X(t,\ldots,t) & \text{propositional variables} \\
 & \mid & P \supset P \mid P \wedge P \mid P \vee P \mid \top \mid \bot \mid \neg P & \\
 & \mid & \forall v.P \mid \exists v.P & \text{1st-order quantifiers} \\
 & \mid & \forall X.P \mid \exists X.P & \text{2nd-order quantifiers}
\end{array}
$$

- In second order predicate logic all other connectives are definable via implication and universal quantification.

$$
\begin{array}{rcl}
\bot & \equiv & \forall X.X \\
P_1 \wedge P_2 & \equiv & \forall X.(P_1 \supset P_2 \supset X) \supset X \\
P_1 \vee P_2 & \equiv & \forall X.(P_1 \supset X) \supset (P_2 \supset X) \supset X \\
\exists v.P & \equiv & \forall X.(\forall v.P \supset X) \supset X \\
\exists X.P & \equiv & \forall Y.(\forall X.P \supset Y) \supset Y
\end{array}
$$