MTAT.05.105 Type Theory

Type inference

Type inference

- In Church-style λ-calculi, λ-terms contain sufficent explicit type annotations making type checking and type inference easy.
- Curry-style systems keep untyped syntax for terms and use types as well-formedness predicate.
- In general, for Curry-style type systems the type inference (and also type checking) is undecidable.
  - Second-order λ-calculus.
- But, for some simpler systems type inference is possible.
  - Simply typed λ-calculus.
  - Hindley-Milner polymorhism.

Simply typed λ-calculus a'la Curry

- Types (the same as in $\lambda{\rightarrow}$ a'la Church):

$$\begin{array}{rcll} \tau & ::= & \alpha & \text{type variable} \\ & | & \tau_1 \rightarrow \tau_2 & \text{function type} \end{array}$$

- Terms (the same as in pure λ-calculus):

$$\begin{array}{rcll} e & ::= & x & \text{variable} \\ & | & e_1\, e_2 & \text{application} \\ & | & \lambda x.\, e & \text{abstraction} \end{array}$$

- Typing rules:

$$\frac{}{\Gamma, x : \tau \vdash x : \tau} \qquad \frac{\Gamma, x : \sigma \vdash e : \tau}{\Gamma \vdash \lambda x.\, e : \sigma \rightarrow \tau}$$

$$\frac{\Gamma \vdash e_1 : \sigma \rightarrow \tau \quad \Gamma \vdash e_2 : \sigma}{\Gamma \vdash e_1\, e_2 : \tau}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

- Notation:
  - $S, S', \ldots$ type substitutions
  - $\tau \succ \tau' \iff \exists S\,[\tau' = S(\tau)]$;
  - $\Gamma \succ \Gamma' \iff \exists S\,[\Gamma' \supseteq S(\Gamma)]$.
- Definition: (Γ, τ) is a principal pair for the term e iff

(i) $\Gamma \vdash e : \tau$;

(ii) $\Gamma' \vdash e : \tau' \iff \Gamma \succ \Gamma' \;\wedge\; \tau \succ \tau'$.

- For a closed term e, the type τ in the principal pair $(\emptyset, \tau)$ is called a principal type.
- Theorem: Any typable term e has a correponding principal pair (Γ, τ). Moreover, the pair is unique up to the renaming of type variables.

Type inference for $\lambda\!\to$ a'la Curry

Type inference algorithm:

- Annotate every subterm and bounding occurence of a variable by a unique type variable.
- Generate a constraint system using a following set of rules:

$$\frac{x^\alpha \in \Gamma}{\Gamma \vdash x^\beta \Rightarrow \{\alpha = \beta\}} \qquad \frac{\Gamma, x^\alpha \vdash e^\beta \Rightarrow E}{\Gamma \vdash (\lambda x^\alpha .\, e^\beta)^\gamma \Rightarrow \{\gamma = \alpha \to \beta\} \cup E}$$

$$\frac{\Gamma \vdash e_1^\alpha \Rightarrow E_1 \qquad \Gamma \vdash e_2^\beta \Rightarrow E_2}{\Gamma \vdash (e_1^\alpha\, e_2^\beta)^\gamma \Rightarrow \{\alpha = \beta \to \gamma\} \cup E_1 \cup E_2}$$

- Solve the constraint system by finding the most general unifier.
  - If it doesn't exist $\Longrightarrow$ the term is not typable.

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\frac{}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}$$

Type inference for $\lambda\!\to$ a'la Curry

$$\dfrac{\dfrac{}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\cfrac{\cfrac{\cfrac{}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}$$

Type inference for $\lambda\!\to$ a'la Curry

$$\cfrac{\cfrac{\cfrac{\cfrac{}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma \vdash x^{\alpha_4}} \qquad \cfrac{}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{}{\Gamma \vdash x^{\alpha_4}} \qquad \dfrac{\dfrac{}{\Gamma \vdash y^{\alpha_5}} \qquad \dfrac{}{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}}}}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
  \dfrac{
    \dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \;\Rightarrow\; E_1}
    \qquad
    \dfrac{\overline{\Gamma \vdash y^{\alpha_5}} \qquad \overline{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}
  }{
    \dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}{
      \dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}{
        x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}
      }
    }
  }
}{
  \vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}
}
$$

Constraints:

$$E_1 \quad = \quad \{\alpha_1 = \alpha_4\}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{}{\Gamma \vdash z^{\alpha_6}}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}}{}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} \\
E_2 &= \{\alpha_2 = \alpha_5\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
  \dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1}
  \qquad
  \dfrac{
    \dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2}
    \quad
    \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}
  }{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1}}
}{
  \dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}{
    \dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}{
      \dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}{
        \vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}
      }
    }
  }
}
$$

Constraints:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_4\} \\
E_2 & = & \{\alpha_2 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_6\}
\end{array}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \qquad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2}}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} \\
E_2 &= \{\alpha_2 = \alpha_5\} \\
E_3 &= \{\alpha_3 = \alpha_6\} \\
E_4 &= \{\alpha_5 = \alpha_6 \to \beta_1\} \\
&\quad \cup \; E_2 \cup E_3
\end{aligned}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{
\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1}
\qquad
\dfrac{
\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2}
\quad
\dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}
}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}
}{
\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{
\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}}{
\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}{
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}
}}}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} \\
E_2 &= \{\alpha_2 = \alpha_5\} \\
E_3 &= \{\alpha_3 = \alpha_6\} \\
E_4 &= \{\alpha_5 = \alpha_6 \to \beta_1\} \\
&\quad \cup \; E_2 \cup E_3
\end{aligned}
$$

$$
E_5 = \{\alpha_4 = \beta_1 \to \beta_2\} \cup E_1 \cup E_4
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1}
\qquad
\dfrac{
\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2}
\quad
\dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}
}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}
}{
\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{
\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{
\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}}{
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}
}}}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} \\
E_2 &= \{\alpha_2 = \alpha_5\} \\
E_3 &= \{\alpha_3 = \alpha_6\} \\
E_4 &= \{\alpha_5 = \alpha_6 \to \beta_1\} \\
&\quad \cup \ E_2 \cup E_3
\end{aligned}
\qquad
\begin{aligned}
E_5 &= \{\alpha_4 = \beta_1 \to \beta_2\} \cup E_1 \cup E_4 \\
E_6 &= \{\gamma_1 = \alpha_3 \to \beta_2\} \cup E_5
\end{aligned}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1}\in\Gamma}{\Gamma\vdash x^{\alpha_4}\Rightarrow E_1}\quad\dfrac{\dfrac{y^{\alpha_2}\in\Gamma}{\Gamma\vdash y^{\alpha_5}\Rightarrow E_2}\quad\dfrac{z^{\alpha_3}\in\Gamma}{\Gamma\vdash z^{\alpha_6}\Rightarrow E_3}}{\Gamma\vdash (y^{\alpha_5}z^{\alpha_6})^{\beta_1}\Rightarrow E_4}}{x^{\alpha_1},y^{\alpha_2},z^{\alpha_3}\vdash (x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2}\Rightarrow E_5}}{x^{\alpha_1},y^{\alpha_2}\vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1}\Rightarrow E_6}}{x^{\alpha_1}\vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2}\Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5}z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} \\
E_2 &= \{\alpha_2 = \alpha_5\} \\
E_3 &= \{\alpha_3 = \alpha_6\} \\
E_4 &= \{\alpha_5 = \alpha_6 \to \beta_1\} \\
&\quad \cup\ E_2 \cup E_3
\end{aligned}
\qquad
\begin{aligned}
E_5 &= \{\alpha_4 = \beta_1 \to \beta_2\} \cup E_1 \cup E_4 \\
E_6 &= \{\gamma_1 = \alpha_3 \to \beta_2\} \cup E_5 \\
E_7 &= \{\gamma_2 = \alpha_2 \to \gamma_1\} \cup E_6
\end{aligned}
$$

Type inference for $\lambda\!\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_4\} & E_5 &= \{\alpha_4 = \beta_1 \to \beta_2\} \cup E_1 \cup E_4 \\
E_2 &= \{\alpha_2 = \alpha_5\} & E_6 &= \{\gamma_1 = \alpha_3 \to \beta_2\} \cup E_5 \\
E_3 &= \{\alpha_3 = \alpha_6\} & E_7 &= \{\gamma_2 = \alpha_2 \to \gamma_1\} \cup E_6 \\
E_4 &= \{\alpha_5 = \alpha_6 \to \beta_1\} & E_8 &= \{\gamma_3 = \alpha_1 \to \gamma_2\} \cup E_7 \\
&\quad \cup \; E_2 \cup E_3 & &
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_3 \to \beta_2, \\
& \gamma_2 = \alpha_2 \to \gamma_1, \\
& \gamma_3 = \alpha_1 \to \gamma_2\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad \{ & \alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_3 \to \beta_2, \\
& \gamma_2 = \alpha_2 \to \gamma_1, \\
& \gamma_3 = \alpha_4 \to \gamma_2 \}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}{}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_3 \to \beta_2, \\
& \gamma_2 = \alpha_5 \to \gamma_1, \\
& \gamma_3 = \alpha_4 \to \gamma_2\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}$$

Constraints:

$$\begin{aligned} E_8 \quad = \quad \{ & \alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\ & \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\ & \gamma_1 = \alpha_6 \to \beta_2, \\ & \gamma_2 = \alpha_5 \to \gamma_1, \\ & \gamma_3 = \alpha_4 \to \gamma_2 \} \end{aligned}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \quad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}$$

Constraints:

$$\begin{aligned} E_8 \quad = \quad \{ & \alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\ & \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\ & \gamma_1 = \alpha_6 \to \beta_2, \\ & \gamma_2 = (\alpha_6 \to \beta_1) \to \gamma_1, \\ & \gamma_3 = \alpha_4 \to \gamma_2 \} \end{aligned}$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{
\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1}
\quad
\dfrac{
\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2}
\quad
\dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}
}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}
}{
\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{
\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{
\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{
\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad & \{\alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_6 \to \beta_2, \\
& \gamma_2 = (\alpha_6 \to \beta_1) \to \gamma_1, \\
& \gamma_3 = (\beta_1 \to \beta_2) \to \gamma_2\}
\end{aligned}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad \{ & \alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_6 \to \beta_2, \\
& \gamma_2 = (\alpha_6 \to \beta_1) \to \alpha_6 \to \beta_2, \\
& \gamma_3 = (\beta_1 \to \beta_2) \to \gamma_2 \}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{\dfrac{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}{\dfrac{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}{\dfrac{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}}}}
$$

Constraints:

$$
\begin{aligned}
E_8 \quad = \quad \{ & \alpha_1 = \alpha_4,\ \alpha_2 = \alpha_5,\ \alpha_3 = \alpha_6, \\
& \alpha_5 = \alpha_6 \to \beta_1,\ \alpha_4 = \beta_1 \to \beta_2, \\
& \gamma_1 = \alpha_6 \to \beta_2, \\
& \gamma_2 = (\alpha_6 \to \beta_1) \to \alpha_6 \to \beta_2, \\
& \gamma_3 = (\beta_1 \to \beta_2) \to (\alpha_6 \to \beta_1) \to \alpha_6 \to \beta_2 \}
\end{aligned}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\frac{\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \Gamma}{\Gamma \vdash x^{\alpha_4} \Rightarrow E_1} \qquad \dfrac{\dfrac{y^{\alpha_2} \in \Gamma}{\Gamma \vdash y^{\alpha_5} \Rightarrow E_2} \quad \dfrac{z^{\alpha_3} \in \Gamma}{\Gamma \vdash z^{\alpha_6} \Rightarrow E_3}}{\Gamma \vdash (y^{\alpha_5} z^{\alpha_6})^{\beta_1} \Rightarrow E_4}}{x^{\alpha_1}, y^{\alpha_2}, z^{\alpha_3} \vdash (x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2} \Rightarrow E_5}}{x^{\alpha_1}, y^{\alpha_2} \vdash (\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1} \Rightarrow E_6}}{x^{\alpha_1} \vdash (\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2} \Rightarrow E_7}}{\vdash (\lambda x^{\alpha_1}.(\lambda y^{\alpha_2}.(\lambda z^{\alpha_3}.(x^{\alpha_4}(y^{\alpha_5} z^{\alpha_6})^{\beta_1})^{\beta_2})^{\gamma_1})^{\gamma_2})^{\gamma_3} \Rightarrow E_8}
$$

Inferred type:

$$\gamma_3 \quad = \quad (\beta \to \gamma) \to (\alpha \to \beta) \to \alpha \to \gamma$$

Solving equations by unification

- Equations can be solved by the repeated application of simplification rules.
- The two main rules are:
  - replace an equation of the form $\tau_1 \to \tau_2 = \tau_3 \to \tau_4$ with two equations $\tau_1 = \tau_3$ and $\tau_2 = \tau_4$;
  - suppose, there is an equation of the form $\alpha = \tau$. If $\alpha \in \mathrm{FV}(\tau)$ then report an error, otherwise substitute τ for α in all equations.
- Auxiliary rules:
  - remove equations of the form $\alpha = \alpha$, $\mathsf{Bool} = \mathsf{Bool}$, etc.;
  - replace $\tau = \alpha$ with $\alpha = \tau$;
  - if there is an equation $\tau_1 = \tau_2$ where the head type constructor differs on each side (eg. $\mathsf{Bool} = \alpha_1 \to \alpha_2$) then report an error.

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\frac{}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Type inference for $\lambda\!\to$ a'la Curry

$$\dfrac{\dfrac{}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}} \qquad \dfrac{}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{}{x^{\alpha_1} \vdash x^{\alpha_2}}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}}
\qquad
\dfrac{\dfrac{}{x^{\alpha_4} \vdash x^{\alpha_5}}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}
}{
\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}
}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6}}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}$$

Constraints:

$$\begin{array}{lcl} E_1 & = & \{\alpha_1 = \alpha_2\} \\ E_2 & = & \{\alpha_4 = \alpha_5\} \end{array}$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}
\qquad
\dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}
}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7}}
$$

Constraints:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_4 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_1 \to \alpha_2\} \cup E_1 \\
E_4 & = & \{\alpha_6 = \alpha_4 \to \alpha_5\} \cup E_2
\end{array}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}
\qquad
\dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}
}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_4 = \alpha_5\} \\
E_3 & = & \{\alpha_3 = \alpha_1 \to \alpha_2\} \cup E_1 \\
E_4 & = & \{\alpha_6 = \alpha_4 \to \alpha_5\} \cup E_2 \\
E_5 & = & \{\alpha_3 = \alpha_6 \to \alpha_7\} \cup E_3 \cup E_4
\end{array}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{aligned}
E_5 \quad = \quad & \{\alpha_1 = \alpha_2,\ \alpha_4 = \alpha_5, \\
& \alpha_3 = \alpha_1 \to \alpha_2, \\
& \alpha_6 = \alpha_4 \to \alpha_5, \\
& \alpha_3 = \alpha_6 \to \alpha_7\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{aligned}
E_5 \quad = \quad \{ \qquad\qquad & \alpha_4 = \alpha_5, \\
\alpha_3 = \alpha_2 \to{} & \alpha_2, \\
\alpha_6 = \alpha_4 \to{} & \alpha_5, \\
\alpha_3 = \alpha_6 \to{} & \alpha_7\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3}}{} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{aligned}
E_5 \quad = \quad \{ & \\
& \alpha_3 = \alpha_2 \rightarrow \alpha_2, \\
& \alpha_6 = \alpha_5 \rightarrow \alpha_5, \\
& \alpha_3 = \alpha_6 \rightarrow \alpha_7\}
\end{aligned}
$$

Type inference for $\lambda\!\to$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}$$

Constraints:

$$\begin{array}{rcl} E_5 & = & \{ \\ & & \quad \alpha_2 = \alpha_6, \\ & & \quad \alpha_6 = \alpha_5 \to \alpha_5, \\ & & \quad \alpha_2 = \alpha_7\} \end{array}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{aligned}
E_5 \quad = \quad \{ & \\
& \alpha_2 = \alpha_5 \to \alpha_5, \\
& \alpha_2 = \alpha_7\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1}}{\vdash (\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3} \Rightarrow E_3} \qquad \dfrac{\dfrac{x^{\alpha_4} \in \{x^{\alpha_4}\}}{x^{\alpha_4} \vdash x^{\alpha_5} \Rightarrow E_2}}{\vdash (\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6} \Rightarrow E_4}}{\vdash ((\lambda x^{\alpha_1}.x^{\alpha_2})^{\alpha_3}(\lambda x^{\alpha_4}.x^{\alpha_5})^{\alpha_6})^{\alpha_7} \Rightarrow E_5}
$$

Constraints:

$$
\begin{array}{lcl}
E_5 & = & \{ \\
 & & \quad \alpha_7 = \alpha_5 \rightarrow \alpha_5 \\
 & & \}
\end{array}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\frac{}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\cfrac{\cfrac{}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{}{x^{\alpha_1} \vdash x^{\alpha_2}} \qquad \dfrac{}{x^{\alpha_1} \vdash x^{\alpha_3}}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$
\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}}
$$

Constraints:

$$
\begin{aligned}
E_1 &= \{\alpha_1 = \alpha_2\} \\
E_2 &= \{\alpha_1 = \alpha_3\}
\end{aligned}
$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5}}$$

Constraints:

$$\begin{aligned} E_1 &= \{\alpha_1 = \alpha_2\} \\ E_2 &= \{\alpha_1 = \alpha_3\} \\ E_3 &= \{\alpha_2 = \alpha_3 \to \alpha_4\} \cup E_1 \cup E_2 \end{aligned}$$

Type inference for $\lambda\!\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}}$$

Constraints:

$$\begin{array}{lcl}
E_1 & = & \{\alpha_1 = \alpha_2\} \\
E_2 & = & \{\alpha_1 = \alpha_3\} \\
E_3 & = & \{\alpha_2 = \alpha_3 \to \alpha_4\} \cup E_1 \cup E_2 \\
E_4 & = & \{\alpha_5 = \alpha_1 \to \alpha_4\} \cup E_3
\end{array}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}}$$

Constraints:

$$\begin{aligned} E_4 \quad = \quad & \{\alpha_1 = \alpha_2,\ \alpha_1 = \alpha_3, \\ & \ \alpha_2 = \alpha_3 \rightarrow \alpha_4, \\ & \ \alpha_5 = \alpha_1 \rightarrow \alpha_4 \\ & \} \end{aligned}$$

Type inference for $\lambda\!\!\to$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}}$$

Constraints:

$$E_4 = \{ \quad \alpha_2 = \alpha_3, \quad \alpha_2 = \alpha_3 \to \alpha_4, \quad \alpha_5 = \alpha_2 \to \alpha_4 \quad \}$$

Type inference for $\lambda\!\to$ a'la Curry

$$\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{\dfrac{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}}$$

Constraints:

$$\begin{array}{lcl} E_4 & = & \{ \\ & & \quad \alpha_3 = \alpha_3 \to \alpha_4, \\ & & \quad \alpha_5 = \alpha_3 \to \alpha_4 \\ & & \} \end{array}$$

Type inference for $\lambda\!\rightarrow$ a'la Curry

$$\dfrac{\dfrac{\dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_2} \Rightarrow E_1} \qquad \dfrac{x^{\alpha_1} \in \{x^{\alpha_1}\}}{x^{\alpha_1} \vdash x^{\alpha_3} \Rightarrow E_2}}{x^{\alpha_1} \vdash (x^{\alpha_2} x^{\alpha_3})^{\alpha_4} \Rightarrow E_3}}{\vdash (\lambda x^{\alpha_1}.(x^{\alpha_2} x^{\alpha_3})^{\alpha_4})^{\alpha_5} \Rightarrow E_4}$$

Constraints:

$$E_4 = \{ \; \alpha_3 = \alpha_3 \to \alpha_4, \; \alpha_5 = \alpha_3 \to \alpha_4 \; \}$$

Error!

Hindley-Milner polymorhism

- Type inference (and type checking) for Curry-style second-order λ-calculus is undecidable.
  - Also, the prinicipal typing property doesn't hold.
- Hindley-Milner type system is a restricted version of Curry-style $\lambda 2$, where universal quantification is allowed only in the "top-level".
- Uses special syntactic construct (let-expressions) for defining variables with polymorhic type (sc. let-polymorhism).
  - Function parameters (ie. λ-bound variables) are monomorphic.
- Type inference for Hindley-Milner type system is decidable.
- Note: In ML/Haskell, the top-level universal quantification is implicit.

Hindley-Milner polymorhism

- Types and type schemes:

τ	$::=$	α	type variable
	$\mid$	$\tau_1 \to \tau_2$	function type
σ	$::=$	τ	monomorphic type
	$\mid$	$\forall \alpha . \sigma$	polymorphic type

- Terms:

e	$::=$	x	variable
	$\mid$	$e_1\, e_2$	application
	$\mid$	$\lambda x.\, e$	abstraction
	$\mid$	$\mathbf{let}\ x = e_1\ \mathbf{in}\ e_0$	let-expression

- Reduction rules:

$$\begin{array}{lcl} (\lambda x . e_0)\, e_1 & \to & e_0[x \mapsto e_1] \\ \mathbf{let}\ x = e_1\ \mathbf{in}\ e_0 & \to & e_0[x \mapsto e_1] \end{array}$$

Hindley-Milner polymorhism

- Typing rules:

$$\frac{}{\Gamma, x : \sigma \vdash x : \sigma}$$

$$\frac{\Gamma \vdash e_1 : \sigma \quad \Gamma, \{x{:}\sigma\} \vdash e_0 : \tau}{\Gamma \vdash \textbf{let } x = e_1 \textbf{ in } e_0 : \tau}$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_0}{\Gamma \vdash \lambda x.e : \tau_1 \rightarrow \tau_0}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \rightarrow \tau_0 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\, e_2 : \tau_0}$$

$$\frac{\Gamma \vdash e : \sigma}{\Gamma \vdash e : \forall \alpha.\sigma} \; (\alpha \notin \mathrm{FV}(\Gamma))$$

$$\frac{\Gamma \vdash e : \forall \alpha.\sigma}{\Gamma \vdash e : \sigma[\tau \mapsto \alpha]}$$

Hindley-Milner polymorhism

$$\frac{}{\Gamma \vdash \textbf{let } id = \lambda x.x \textbf{ in } (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{}{\Gamma \vdash \lambda x.x : \forall\alpha.\alpha \to \alpha} \qquad \dfrac{}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ \ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ \ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}}{\Gamma \vdash \lambda x.x : \forall\alpha.\alpha \to \alpha} \qquad \dfrac{}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}}{\Gamma \vdash \lambda x.x : \forall \alpha.\alpha \to \alpha} \qquad \dfrac{}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \textbf{let}\ id = \lambda x.x\ \textbf{in}\ (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}}{\Gamma \vdash \lambda x.x : \forall \alpha.\alpha \to \alpha} \quad \dfrac{\dfrac{}{\Gamma' \vdash id\,3 : \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

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\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}
}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}
}{\Gamma \vdash \lambda x.x : \forall\alpha.\alpha \to \alpha}
\quad
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\dfrac{}{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}}
\quad
\dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}
}{\Gamma' \vdash id\,3 : \mathtt{I}}
\quad
\dfrac{}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}
}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}
}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}
$$

Hindley-Milner polymorhism

$$
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      \dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}
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      \Gamma \vdash \lambda x.x : \alpha \to \alpha
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    \Gamma \vdash \lambda x.x : \forall \alpha.\alpha \to \alpha
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  \qquad
  \dfrac{
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      \dfrac{
        \dfrac{}{\Gamma' \vdash id : \forall \alpha.\alpha \to \alpha}
      }{
        \Gamma' \vdash id : \mathtt{I} \to \mathtt{I}
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      \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}
    }{
      \Gamma' \vdash id\,3 : \mathtt{I}
    }
    \qquad
    \dfrac{}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}
  }{
    \Gamma, id{:}\forall \alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}
  }
}{
  \Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}
}
$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}}{\Gamma \vdash \lambda x.x : \forall \alpha.\alpha \to \alpha} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \forall \alpha.\alpha \to \alpha}}{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}}{\Gamma' \vdash id\,3 : \mathtt{I}} \quad \dfrac{\dfrac{}{\Gamma' \vdash id : \mathtt{B} \to \mathtt{B}} \quad \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}}{\Gamma, id{:}\forall \alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

$$\Gamma \vdash \textbf{let}\ id = \lambda x.x\ \textbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}$$

Hindley-Milner polymorhism

$$
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\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}}{\Gamma \vdash \lambda x.x : \forall \alpha.\alpha \to \alpha}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \forall \alpha.\alpha \to \alpha}}{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}}{\Gamma' \vdash id\, 3 : \mathtt{I}}
\qquad
\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \forall \alpha.\alpha \to \alpha}}{\Gamma' \vdash id : \mathtt{B} \to \mathtt{B}} \quad \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}}{\Gamma' \vdash id\, \mathtt{T} : \mathtt{B}}
}{\Gamma, id{:}\forall \alpha.\alpha \to \alpha \vdash (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}
}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}
$$

Hindley-Milner polymorhism

$$
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  \dfrac{
    \dfrac{
      \dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}
    }{\Gamma \vdash \lambda x.x : \alpha \to \alpha}
  }{\Gamma \vdash \lambda x.x : \forall\alpha.\alpha \to \alpha}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma' \vdash id : \forall\alpha.\alpha \to \alpha}
      }{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}}
      \quad
      \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}
    }{\Gamma' \vdash id\, 3 : \mathtt{I}}
    \quad
    \dfrac{
      \dfrac{
        \dfrac{}{\Gamma' \vdash id : \forall\alpha.\alpha \to \alpha}
      }{\Gamma' \vdash id : \mathtt{B} \to \mathtt{B}}
      \quad
      \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}
    }{\Gamma' \vdash id\, \mathtt{T} : \mathtt{B}}
  }{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}
}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\, 3,\ id\, \mathtt{T}) : \mathtt{I} \times \mathtt{B}}
$$

Note that the following is not derivable:

$$\Gamma \vdash (\lambda id.(id\, 3,\ id\, \mathtt{T}))\,(\lambda x.x) : \mathtt{I} \times \mathtt{B}$$

Hindley-Milner polymorhism

- Syntax-directed typing rules:

$$\frac{\sigma \succ \tau}{\Gamma, x : \sigma \vdash x : \tau} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma, \{x{:}\forall\bar{\alpha}.\tau_1\} \vdash e_0 : \tau_0}{\Gamma \vdash \mathbf{let}\ x = e_1\ \mathbf{in}\ e_0 : \tau_0}\ (\bar{\alpha} \notin \mathrm{FV}(\Gamma))$$

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_0}{\Gamma \vdash \lambda x.e : \tau_1 \to \tau_0} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \to \tau_0 \quad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\, e_2 : \tau_0}$$

Hindley-Milner polymorhism

$$\frac{}{\Gamma \vdash \textbf{let } id = \lambda x.x \textbf{ in } (id\,3,\ id\,\texttt{T}) : \texttt{I} \times \texttt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{}{\Gamma \vdash \lambda x.x : \alpha \rightarrow \alpha} \qquad \dfrac{}{\Gamma, id{:}\forall\alpha.\alpha \rightarrow \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \textbf{let}\ id = \lambda x.x\ \textbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\overline{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha} \qquad \overline{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ \ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ \ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$
\dfrac{\dfrac{\overline{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha}
\qquad
\dfrac{\dfrac{}{\Gamma' \vdash id\,3 : \mathtt{I}} \qquad \dfrac{}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}
{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}
$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}}{\Gamma' \vdash id\,3 : \mathtt{I}} \qquad \dfrac{}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

$$\dfrac{\dfrac{\dfrac{}{\Gamma, x{:}\alpha \vdash x : \alpha}}{\Gamma \vdash \lambda x.x : \alpha \to \alpha} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma' \vdash id : \mathtt{I} \to \mathtt{I}} \quad \dfrac{}{\Gamma' \vdash 3 : \mathtt{I}}}{\Gamma' \vdash id\,3 : \mathtt{I}} \qquad \dfrac{\dfrac{}{\Gamma' \vdash id : \mathtt{B} \to \mathtt{B}} \quad \dfrac{}{\Gamma' \vdash \mathtt{T} : \mathtt{B}}}{\Gamma' \vdash id\,\mathtt{T} : \mathtt{B}}}{\Gamma, id{:}\forall\alpha.\alpha \to \alpha \vdash (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}}{\Gamma \vdash \mathbf{let}\ id = \lambda x.x\ \mathbf{in}\ (id\,3,\ id\,\mathtt{T}) : \mathtt{I} \times \mathtt{B}}$$

Hindley-Milner polymorhism

- Term with a very complex type:

$$
\begin{array}{l}
\textbf{let}\ pair = \lambda xyz.z\,x\,y\ \textbf{in} \\
\quad \textbf{let}\ x_1 = \lambda y.pair\,y\,y\ \textbf{in} \\
\quad\quad \textbf{let}\ x_2 = \lambda y.x_1(x_1\,y)\ \textbf{in} \\
\quad\quad\quad \textbf{let}\ x_3 = \lambda y.x_2(x_2\,y)\ \textbf{in} \\
\quad\quad\quad\quad \textbf{let}\ x_4 = \lambda y.x_3(x_3\,y)\ \textbf{in} \\
\quad\quad\quad\quad\quad \textbf{let}\ x_5 = \lambda y.x_4(x_4\,y)\ \textbf{in} \\
\quad\quad\quad\quad\quad\quad x_5(\lambda y.y)
\end{array}
$$