Abstract Process Categories

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2 February 2013

Functional reactive programming (FRP)

- extension of functional programming
- allows programmers to deal with temporal aspects in a declarative fashion
- two key features:
  - time-dependent type membership
  - temporal type constructors
- Curry–Howard correspondence to temporal logic:
  - time-dependent trueness
  - temporal operators

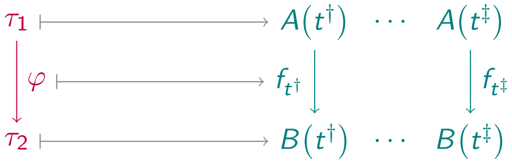
Categorical models of FRP

- ingredients:

  totally ordered set $(T, \leqslant)$ time scale

  CCCC $\mathcal{B}$ simple types and functions

- functor category $\mathcal{B}^T$ models FRP types and FRP operations, with indices denoting inhabitation times:



Temporal type constructors

- process type constructors modeled by functors $\rhd''$ and $\blacktriangleright''$:

$$(A \rhd'' B)(t) = \coprod_{t' \in (t,\infty)} \left(\left(\prod_{t'' \in (t,t')} A(t'') \right) \times B(t') \right)$$

$$(A \blacktriangleright'' B)(t) = (A \rhd'' B)(t) + \prod_{t' \in (t,\infty)} A(t')$$

- variants that also deal with the present:

$$A \rhd' B = A \times A \rhd'' B \qquad A \rhd B = B + A \rhd' B$$

$$A \blacktriangleright' B = A \times A \blacktriangleright'' B \qquad A \blacktriangleright B = B + A \blacktriangleright' B$$

- behaviors and events as special processes:

$$\Box' A = A \blacktriangleright'' 0 \qquad \Box A = A \blacktriangleright' 0$$

$$\Diamond' B = 1 \rhd' B \qquad \Diamond B = 1 \rhd B$$

Topic of this talk

- axiomatically defined categorical semantics for this style of FRP
- road to this semantics:
  1. categorical models of intuitionistic S4 (Kobayashi, 1997; Bierman and de Paiva, 2000)
  2. temporal categories for FRP with behaviors and events (Jeltsch, 2012)
  3. abstract process categories for FRP with arbitrary processes (this talk)

Basic structure

- cartesian closed category $\mathcal{C}$ with coproducts
- process functors:

$$\rhd'', \blacktriangleright'' : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

- unified process functor:

$$- \rhd''_{\_} - : \mathcal{C} \times \mathbf{2} \times \mathcal{C} \to \mathcal{C}$$

- traditional process functors by specialization:

$$\rhd'' = \rhd''_0$$
$$\blacktriangleright'' = \rhd''_1$$

- weakening as mapping:

$$\frac{w : 0 \to 1}{\rhd''_w : \rhd'' \to \blacktriangleright''}$$

Comonads and more

- three kinds of structures:
  - comonads:

$$\varepsilon_{A,W,B} : A \rhd'_W B \to A$$
$$\delta_{A,W,B} : A \rhd'_W B \to (A \rhd'_W B) \rhd'_W B$$

  - ideal comonads:

$$\delta'_{A,W,B} : A \rhd''_W B \to (A \rhd'_W B) \rhd''_W B$$

  - "real comonads":

$$\nu_{A,W,B} : A \rhd_W B \to (A \rhd'_W B) \rhd_W B$$

- derivation:

ideal comonads → comonads → "real comonads"

Monads and more

- three kinds of structures:
  - monads:

$$\eta_{A,W,B} : B \to A \rhd_W B$$
$$\mu_{A,W,B} : A \rhd_W (A \rhd_W B) \to A \rhd_W B$$

  - ideal monads:

$$\mu'_{A,W,B} : A \rhd'_W (A \rhd_W B) \to A \rhd'_W B$$

  - "fantastic monads":

$$\mu''_{A,W,B} : A \rhd''_W (A \rhd_W B) \to A \rhd''_W B$$

- derivation:

"fantastic monads" → ideal monads → monads

Merging

- natural transformation of the following type:

$$A_1 \rhd''_{W_1} B_1 \times A_2 \rhd''_{W_2} B_2$$
$$\downarrow$$
$$(A_1 \times A_2) \rhd''_{W_1 \times W_2} ((A_1, W_1, B_1) \odot (A_2, W_2, B_2))$$

- definition of $\odot$:

$$(A_1, W_1, B_1) \odot (A_2, W_2, B_2)$$
$$=$$
$$(B_1 \times B_2) + \left(B_1 \times A_2 \rhd'_{W_2} B_2\right) + \left(A_1 \rhd'_{W_1} B_1 \times B_2\right)$$

- $W_1 \times W_2$ is minimum of W_1 and W_2
- variants of merging for $\rhd'$ and $\rhd$
- nullary version of merging with the following type:

$$1 \rightarrow 1 \rhd''_1 0$$

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Conclusions and further work

- conclusions:
  - developed abstract process categories (APCs)
  - axiomatically defined categorical semantics for FRP with processes
  - generalize temporal categories (Jeltsch, 2012)
- further work:
  - extensions of APCs:
    - recursion
    - stateful objects
  - FRP implementation with API inspired by (extended versions of) APCs

References

[Kobayashi, 1997]
Satoshi Kobayashi.
Monad as Modality.
Theoretical Computer Science, 175 (1):29–74, 1997.

[Bierman and de Paiva, 2000]
Gavin Bierman and Valeria de Paiva.
On an Intuitionistic Modal Logic.
Studia Logica, 65 (3):383–416, 2000.

[Jeltsch, 2012]
Wolfgang Jeltsch.
Towards a Common Categorical Semantics for Linear-Time Temporal Logic and Functional Reactive Programming.
Electronic Notes in Theoretical Computer Science, 286:229–242, 2012.