A Formalization of the Max-flow Min-cut Theorem in Higher Order Logic

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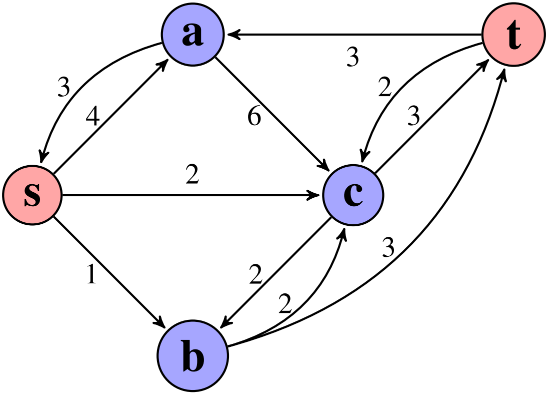
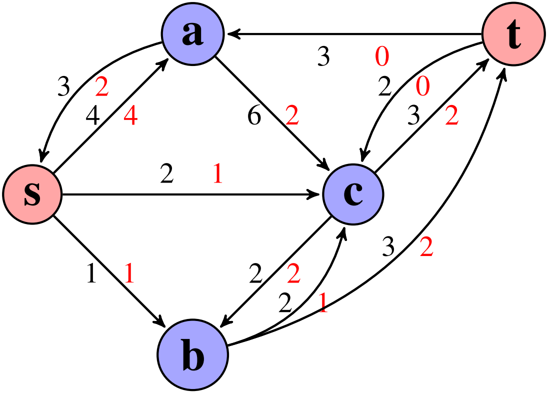
Overview

- Maximum flow problem (MFP): find the maximal value flow for a given network
- The Max-flow Min-cut theorem gives a solution to MFP. It states that the maximum value flow is equal to the minimal s-t cut capacity for the given network.

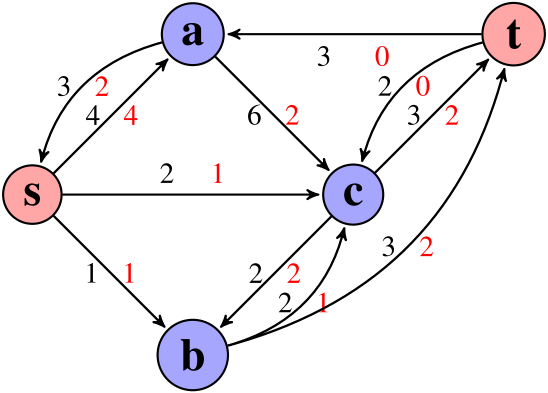
Network



Flow

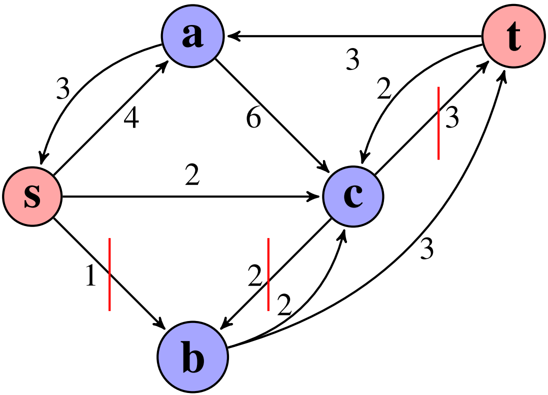


Flow Value



$$v(f) = 4 + 1 + 1 - 2 = 4$$

s-t Cut



$$c(Z) = 1 + 2 + 3 = 6$$

Basic Definitions

There isn't a library on graph in HOL Light so we have to insert every basic graph theoretical definition.

NET

```
⊢ ∀v e s t c.
      NET v e s t c ⇔
      DIR_GRAPH v e ∧
      s IN v ∧
      t IN v ∧
      ¬(s = t) ∧
      (∀u w. u,w IN e ⇒ 0 ≤ c (u,w)) ∧
      (∀u w. ¬(u,w IN e) ⇒ c (u,w) = 0)
```

The Theorem Statement

MAXFLOW_MINCUT

```
⊢ ∀v e s t c.
      NET v e s t c ∧ FINITE v
      ⇒ (∃f z.
              FLOW v e s t c f ∧
              CUT v e s t c z ∧
              FLOW_VALUE e s f = isum z c ∧
              (∀f'. FLOW v e s t c f'
                      ⇒ FLOW_VALUE e s f' ≤
                         FLOW_VALUE e s f) ∧
              (∀z'. CUT v e s t c z'
                      ⇒ isum z c <= isum z' c))
```

Formalization Strategy

(i) Every cut capacity is an upper bound for the set of flow values

(ii) Given a flow f, under a certain condition $\mathcal{S}_f$ we can always generate a flow f' with strictly greater value

$(i) + (ii)$ It exists a flow f^* for which $\mathcal{S}_{f^*}$ doesn't hold

(iii) Verify that the value of f^* is equal to the capacity of some cut

The Condition $\mathcal{S}_f$

Condition $\mathcal{S}_f$:

It exists an undirected walk $\mathcal{P}$ from the source to the target such that, for every edge (u, w) in $\mathcal{P}$, one of the following holds:

- $u\ (u, w)\ w$ occurs in $\mathcal{P}$ and $f(u, w) < c(u, w)$
- $w\ (u, w)\ u$ occurs in $\mathcal{P}$ and $f(u, w) > 0$

The Condition S_f

REACH

```
⊢ (∀v e c f u w.
      REACH c f v e u w [] ⇔
      u = w ∧ w IN v) ∧
  (∀v e c f u w h h1 h2 hs.
      REACH c f v e u w (CONS (h,h1,h2) hs) ⇔
      u IN v ∧
      u = h ∧
      h1,h2 IN e ∧
      (u = h1 ∧
       f (h1,h2) < c (h1,h2) ∧
       REACH c f (v DELETE u) (e DELETE (h1,h2)) h2 w hs ∨
       u = h2 ∧
       0 < f (h1,h2) ∧
       REACH c f (v DELETE u) (e DELETE (h1,h2)) h1 w hs)
```

The Condition $\mathcal{S}_f$

Condition $\mathcal{S}_f$:

It exists an undirected walk $\mathcal{P}$ from the source to the target such that, for every edge (u, w) in $\mathcal{P}$, one of the following holds:

- $u\ (u, w)\ w$ occurs in $\mathcal{P}$ and $f(u, w) < c(u, w)$
- $w\ (u, w)\ u$ occurs in $\mathcal{P}$ and $f(u, w) > 0$

SF

```
⊢ ∀v e s c f.
      SF c f v e s =
      {w | w IN v ∧ (∃hs. REACH c f v e s w hs)}
```

De Bruijn Factor and other facts

- The De Bruijn factor is the quotient of the size of a formalization of a mathematical text and the size of its informal original

  http://www.cs.ru.nl/~freek/factor/

- The De Bruijn factor of the Max-flow Min-cut theorem is 4

- The formalization in HOL Light of the Max-flow Min-cut theorem consists in 3027 lines of source code. More than half are needed to formalize the construction of a greater value flow (ii)

formal-graph-lib

- The complete formal proof of the theorem and much more can be found at

  http://code.google.com/p/formal-graph-lib/