Categorical views on bottom-up tree transducers

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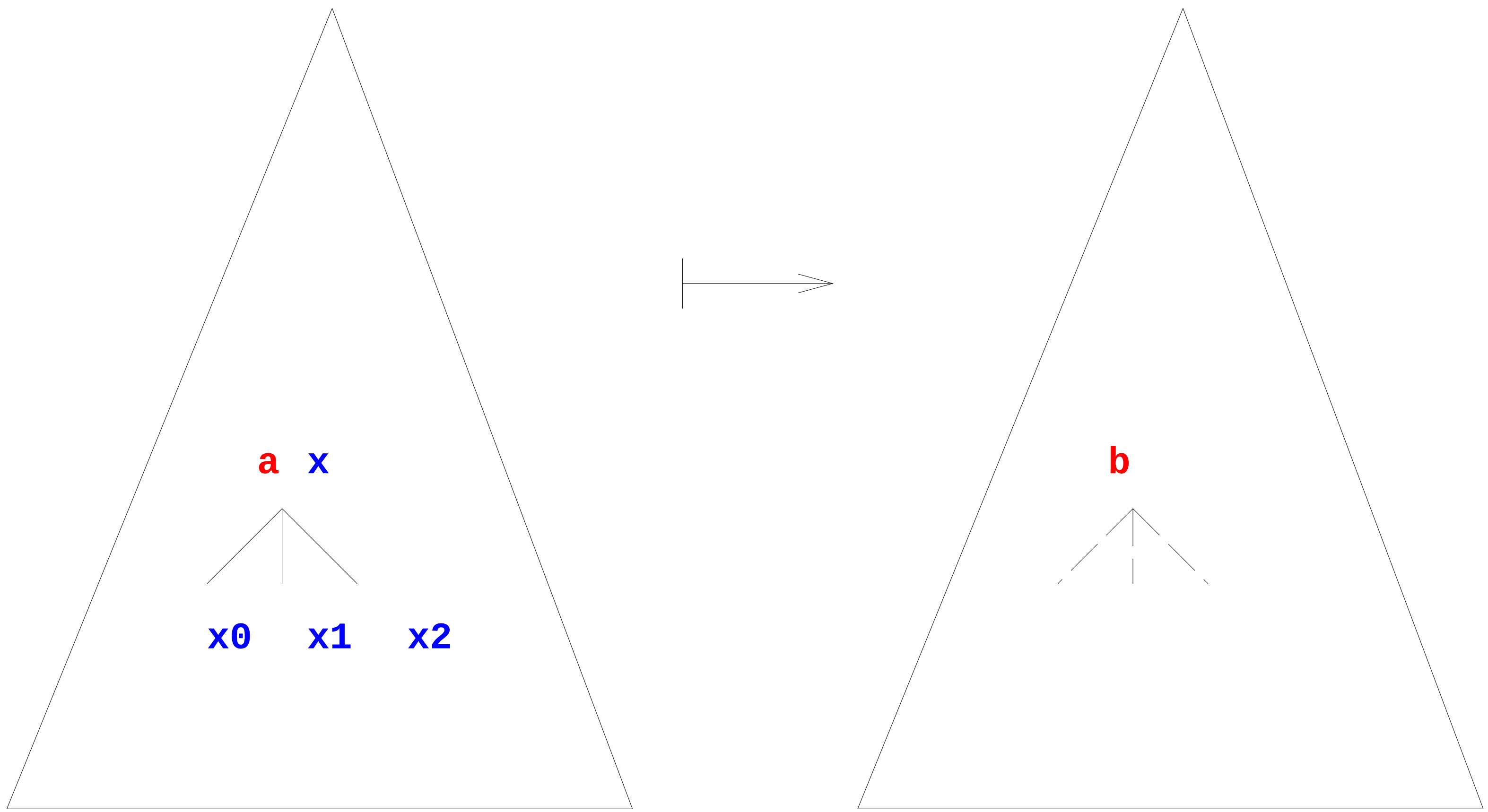
Three types of bottom-up tree transducers / three triangles

- Three types of bottom-up tree transducers, ordered by generality:
  - relabelling (branching-preserving) = purely synthesized attribute grammars
  - rebranching (layering-preserving)
  - 1-n relayering (= the classical notion)
- For each type, we have a triangular picture: transducers (modulo bisimilarity) are the same as (co/bi)-Kleisli maps of a comonad/distributive law and a subclass of tree functions

Relabelling bottom-up tree transducers



RELABELLING BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- F — a fixed endofunctor on the base category
  A, B, C — typical objects of the base category
- $\mathit{LTree}\ A =_{\mathrm{df}} \mu Z. A \times F Z$ — A-labelled F-branching trees
- $DA =_{\mathrm{df}} \mathit{LTree}\ A$ — "subtrees"; D is a comonad on the base category!

realizations

relabelling BU tree trans-s (mod bisim)

$(X, d : A \times FX \to B \times X)$

co-Kleisli behaviors

co-Kleisli maps of D

$k : DA \to B$,

i.e. $k : \mathit{LTree}\ A \to B$

tree function behaviors

relabelling BU tree fun-s

$f : \mathit{LTree}\ A \to \mathit{LTree}\ B$

- We have three different constructed categories on the objects of the base category.
- The three categories are equivalent: the maps are in a 1-1 correspondence, and typing, the identities and composition agree.
- Moreover, for each of the three categories, we have an identity-on-objects inclusion functor from the base category, which preserves products (i.e., an "arrow" and more).
- The "arrows" are equivalent too: the inclusion functors agree as well.

Relabelling bottom-up tree transducers

- Relabelling bottom-up tree transducers for a fixed branching type F are pairs $(X, d : A \times FX \to B \times X)$ (X — state space, d — transition function)
- Identity on A:

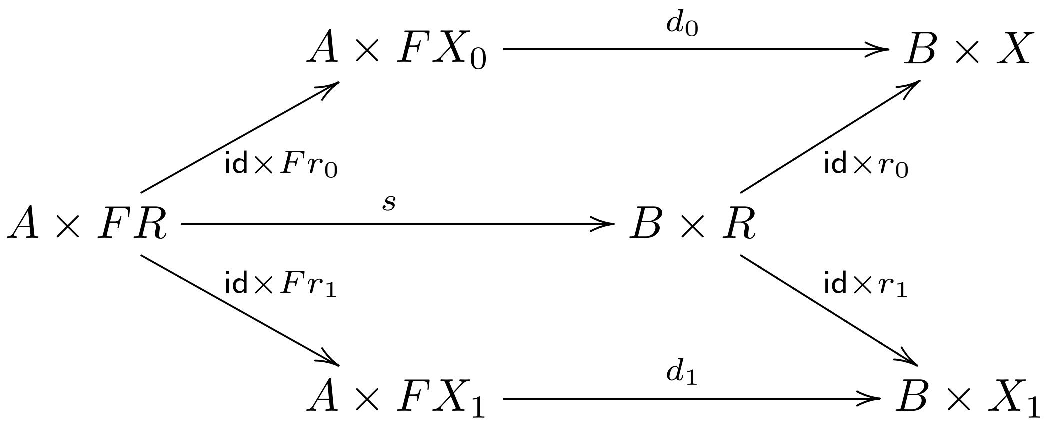
$$(1, A \times F1 \longrightarrow A \longrightarrow A \times 1)$$

- Composition of $(X, d : A \times FX \to B \times X)$ and $(X', e : B \times FX' \to C \times X')$:

$$(X \times X', A \times F(X \times X') \to A \times FX \times FX' \to B \times X \times FX' \to C \times (X \times X')$$

BISIMILARITY OF RELABELLING BU TREE TRANS-S

- $(X_0, d_0 : A \times FX_0 \to B \times X_1)$ and $(X_1, d_1 : A \times FX_1 \to B \times X_1)$ are defined to be bisimilar, if there exist a span (R, r_0, r_1) (a bisimulation) and a map $s : A \times FR \to B \times R$ (its bisimulationhood witness) such that



CO-KLEISLI BEHAVIORS OF RELABELLING BU TREE TRANS-S

- The comonad for relabelling BU tree trans-s is (D, ε, δ) where
  - $DA =_{\mathrm{df}} \mathit{LTree}\, A =_{\mathrm{df}} \mu Z. A \times FZ$ — (sub)trees
  - $\varepsilon_A =_{\mathrm{df}} \ DA \xrightarrow{\cong} A \times F(DA) \xrightarrow{\mathsf{fst}} A$ — extraction of the root label
  - $\delta_A =_{\mathrm{df}} \ DA \xrightarrow{\delta'_A} DA \times D(DA) \xrightarrow{\mathsf{snd}} D(DA)$ — replacement of the label with the subtree at every node
- The map $\delta'_A = \langle \mathsf{id}, \delta_A \rangle$ is given by initiality:

$$
\begin{array}{ccc}
DA & \xrightarrow{\qquad \delta'_A \qquad} & DA \times D(DA) \\
\Big| & & \big| \cong \\
 & & DA \times DA \times F(D(DA)) \\
 & & \big\uparrow \Delta \times \mathsf{id} \\
\Big| \cong & & DA \times F(D(DA)) \\
 & & \big| \cong \\
 & & A \times F(DA) \times F(D(DA)) \\
\Big| & & \big\uparrow \mathsf{id} \times F\mathsf{fst} \times F\mathsf{snd} \\
A \times F(DA) & \xrightarrow{\mathsf{id} \times F\delta'_A} & A \times F(DA \times D(DA))
\end{array}
$$

- Co-Kleisli maps are maps $k : DA \to B$, the identity on A is

$$DA \xrightarrow{\varepsilon_A} A$$

the composition of $k : DA \to B$, $\ell : DB \to C$ is

$$DA \xrightarrow{\delta_A} D(DA) \xrightarrow{Dk} DB \xrightarrow{\ell} C$$

(by the general definition of a co-Kleisli category of a comonad)

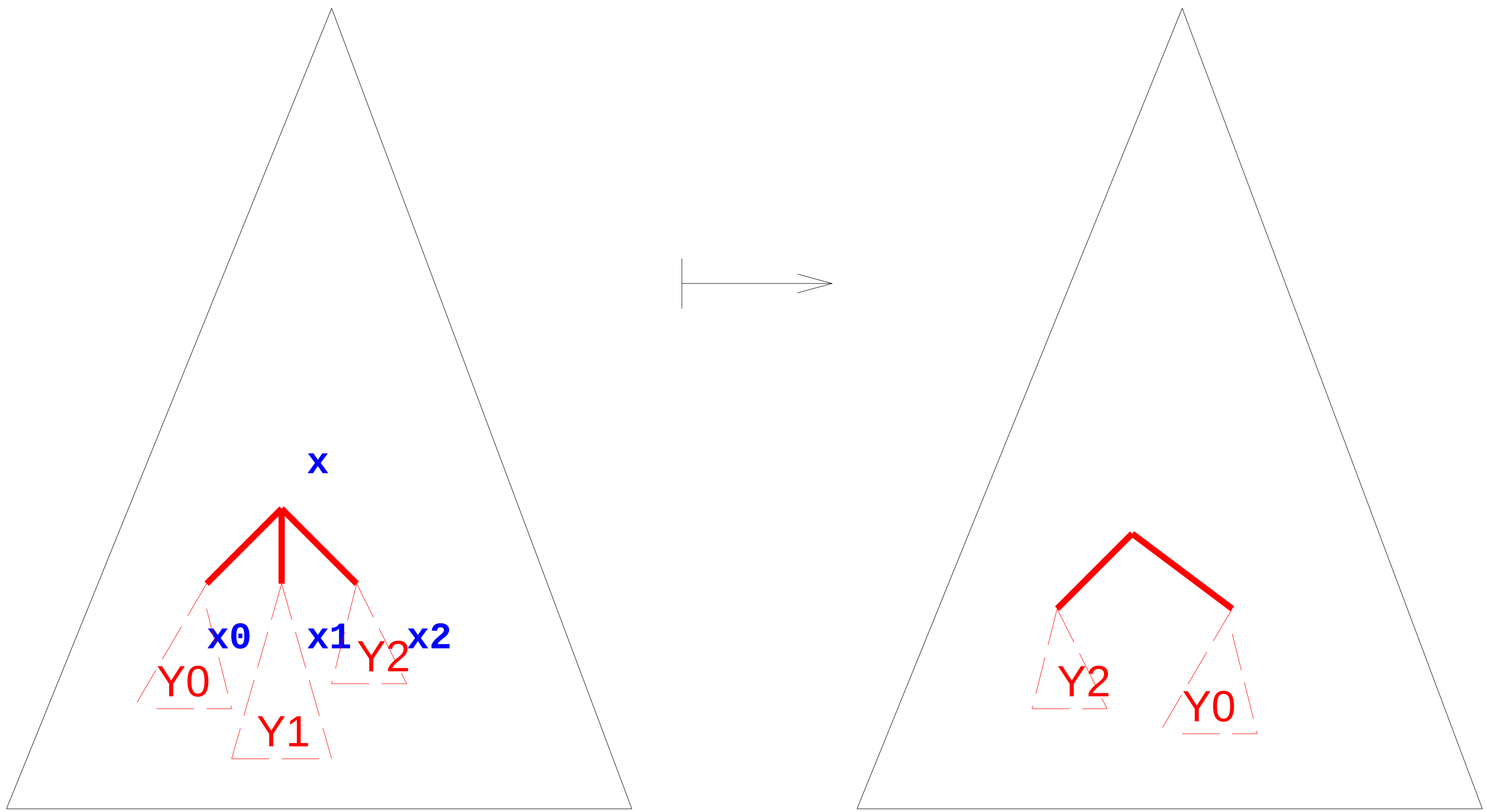
Relabelling BU tree functions

- Tree functions are maps $f : LTree\ A \to LTree\ B$, the identity and composition are taken from the base category.
- A tree function f is defined to be bottom-up relabelling if

$$\begin{array}{ccc}
LTree\ A & \xrightarrow{f} & LTree\ B \\
\big| \cong & & \big| \cong \\
A \times F(LTree\ A) & & B \times F(LTree\ B) \\
\downarrow \mathsf{snd} & & \downarrow \mathsf{snd} \\
F(LTree\ A) & \xrightarrow{Ff} & F(LTree\ B)
\end{array}$$

- The identity tree functions are BU relabelling and the composition of two BU relabelling tree functions is BU relabelling.

Rebranching bottom-up tree transducers



REBRANCHING BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- G, H, K — typical endofunctors on the base category
- $\mathit{Tree}\, G =_{\mathrm{df}} \mu Z.GZ$ — G-branching trees
- $G^\sharp Y =_{\mathrm{df}} G(Y \times \mathit{Tree}\, G)$ – "child-position aware subtrees";
  $(-)^\sharp$ is a comonad on the endofunctor category!

realizations

rebranching BU tree trans-s (mod bisim)

$(X, (d_Y : G(Y \times X) \to HY \times X)_Y)$

co-Kleisli behaviors

co-Kleisli maps of $()^\sharp$

$k : G^\sharp \to H,$

i.e., $(k_Y : G(Y \times \mathit{Tree}\, G) \to HY)_Y$

tree function behaviors

rebranching BU tree fun-s

$f : \mathit{Tree}\, G \to \mathit{Tree}\, H$

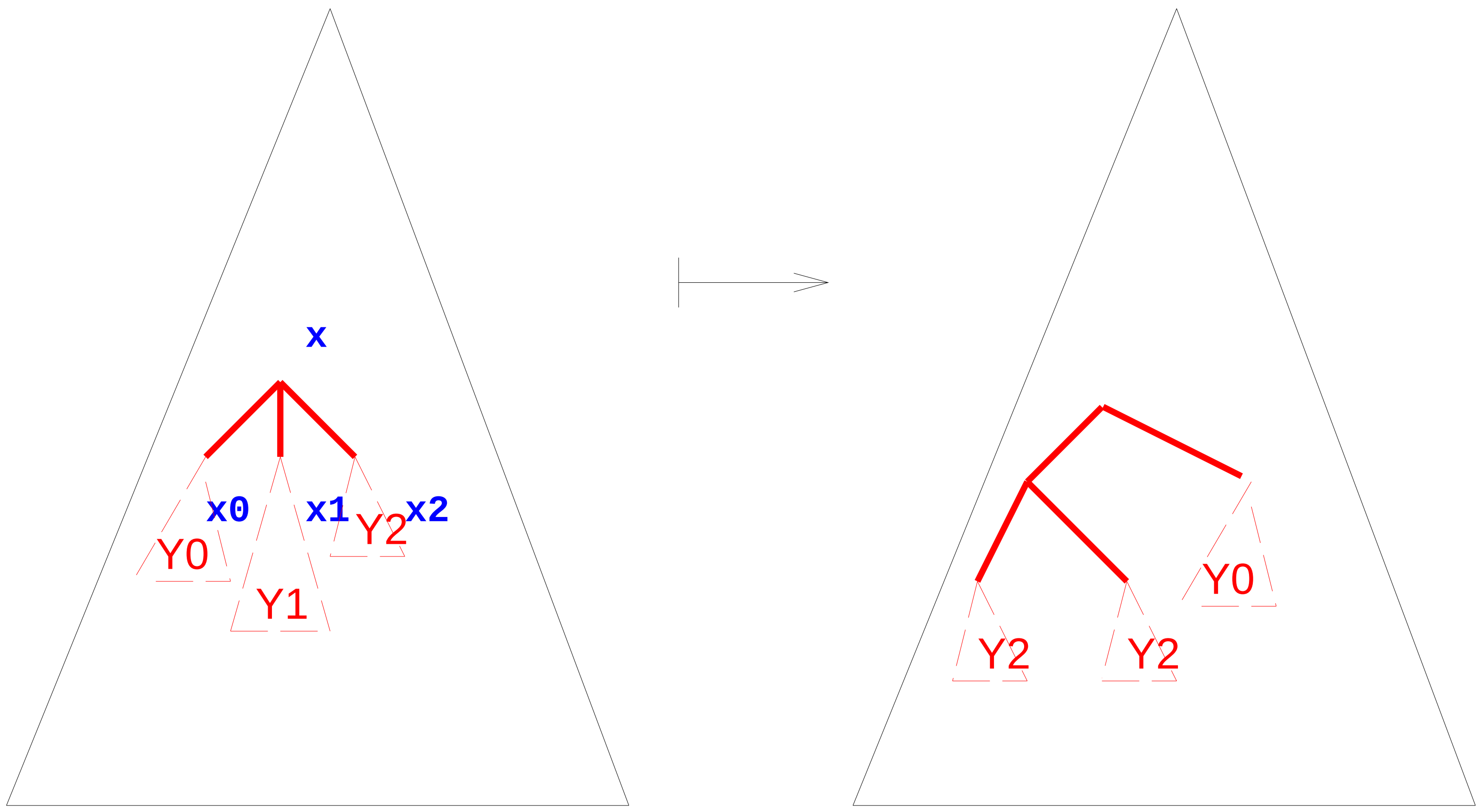
Rebranching BU tree functions

- Tree functions are maps $f : \mathit{Tree}\, G \to \mathit{Tree}\, H$, the identity and composition are taken from the base category.
- A tree function f as above is defined to be rebranching BU if there is a nat transf $(k_Y : G(Y \times \mathit{Tree}\, G) \to HY)_Y$ (its rebranching BU witness) such that

$$
\begin{array}{ccccc}
\mathit{Tree}\, G & & \xrightarrow{\quad f \quad} & & \mathit{Tree}\, H \\
\Big| \cong & & & & \\
G(\mathit{Tree}\, G) & & & & \Big| \cong \\
\Big\downarrow G\Delta & & & & \\
G(\mathit{Tree}\, G \times \mathit{Tree}\, G) & \xrightarrow{k_{\mathit{Tree}\, G}} & H(\mathit{Tree}\, G) & \xrightarrow{H f} & H(\mathit{Tree}\, H)
\end{array}
$$

- k determines f.
- The identity tree functions are relabelling BU and the composition of two relabelling BU tree functions is relabelling BU.

Classical (1-n relayering) bottom-up tree transducers



CLASSICAL (1-N RELAYERING) BOTTOM-UP TREE TRANSDUCERS: THE TRIANGLE

- G, H, K — typical endofunctors on the base category
  $\mathit{Tree}\, G =_{\mathrm{df}} \mu Z.GZ$ — G-branching trees
- $G^\sharp Y =_{\mathrm{df}} G(Y \times \mathit{Tree}\, G)$ — "child-position aware subtrees";
  $(-)^\sharp$ is a comonad on the endofunctor category!
- $G^* Y =_{\mathrm{df}} \mu Z.Y + GZ$ — G-branching trees with Y-leaves;
  $G^* Y$ is a monad on the base category (the free monad of G);
  $(-)^*$ is a monad on the endofunctor category!
- $\mathit{Tree}\, G \cong G^\sharp 0$
- The comonad $(-)^\sharp$ distributes over the monad $()^\star$!

realizations

BU tree trans-s (mod bisim)

$(X, (d_Y : G(Y \times X) \to H^\star Y \times X)_Y)$

bi-Kleisli behaviors

bi-Kleisli maps of $(-)^\sharp$, $(-)^\star$

$k : G^\sharp \to H^\star$,

i.e., $(k_Y : G(Y \times \mathit{Tree}\, G) \to H^\star Y)_Y$

tree function behaviors

BU tree fun-s

$f : \mathit{Tree}\, G \to \mathit{Tree}\, H$

1-N RELAYERING BU TREE FUNCTIONS

- As before, tree functions are maps $f : \mathit{Tree}\, G \to \mathit{Tree}\, H$, the identity and composition are taken from the base category.
- A tree function f as above is defined to be 1-n relayering BU if there is a nat transf $(k_Y : G(Y \times \mathit{Tree}\, G) \to H^\star Y)_Y$ (its rebranching BU witness) such that

$$
\begin{array}{ccccccc}
\mathit{Tree}\, G & & \xrightarrow{\quad f \quad} & & \mathit{Tree}\, H & \overset{\cong}{\text{———}} & H^\star 0 \\
\Big\downarrow{\scriptstyle \cong} & & & & & & \Big\uparrow{\scriptstyle (\mathrm{mult}_H)_0} \\
G(\mathit{Tree}\, G) & & & & & & \\
\Big\downarrow{\scriptstyle G\Delta} & & & & & & \\
G(\mathit{Tree}\, G \times \mathit{Tree}\, G) & \xrightarrow{k_{\mathit{Tree}\, G}} & H^\star(\mathit{Tree}\, G) & \xrightarrow{Hf} & H^\star(\mathit{Tree}\, H) & \overset{\cong}{\text{———}} & H^\star(H^\star 0)
\end{array}
$$

- k determines f.
- The identity tree functions are 1-n relayering BU and the composition of two 1-n relayering BU tree functions is 1-n relayering BU.

VARIATIONS: TOP-DOWN TREE TRANSDUCERS

- The same types of tree transducers are possible, to represent top-down tree functions of these types.
  - relabelling TD TTs:

$$(X, q_I : 1 \to X, d : A \times X \to B \times (F'1 \Rightarrow X))$$

    ($F'1 \Rightarrow X$ — assignments of a state to every child of the current node in the input tree)

  - rebranching TD TTs:

$$(X, q_I : 1 \to X, (d_Y : GY \times X \to H(Y \times X))_Y)$$

  - 1-n relayering TD TTs:

$$(X, q_I : 1 \to X, (d_Y : GY \times X \to H^*(Y \times X))_Y)$$

VARIATIONS: RELABELLING TREE TRANSDUCERS WITH LOOKAHEAD

- Relabelling transducers can be augmented with lookahead, so they can represent functions using information from both below and above any given node.
  - relabelling BU TTs with lookahead:

$$(X, d : A \times (\mu Z.1 + A \times F'1) \times FX \to B \times X)$$

  - relabelling TD TTs with lookahead:

$$(X, q_I : 1 \to X, d : \mathit{LTree}\, A \times X \to B \times (F'1 \Rightarrow X))$$