Wrap-Around Sliding-Window Near-ML Decoding of Binary LDPC Codes Over the BEC

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***Abstract*—A novel method of low-complexity near-maximum-likelihood (ML) decoding of quasi-cyclic (QC) low-density parity-check (LDPC) codes over the binary erasure channel is presented. The idea is similar to wrap-around decoding of tail-biting convolutional codes. ML decoding is applied to a relatively short window which is cyclically shifted along the received sequence. The procedure is repeated until either all erasures have been corrected, or no new erasures are corrected at a certain round. A new upper bound on the ensemble-average ML decoding error probability for a finite-length row-regular LDPC code family is derived and presented. Furthermore, a few examples of regular and irregular QC LDPC codes are studied by simulations and their performance is compared with the ensemble-average performance. Finally, the impact of the codeword weight and stopping set size spectra on the ML and belief-propagation decoding performance is discussed.**

I. Introduction

Although asymptotic ensembles of low-density parity-check (LDPC) codes are provably capacity-achieving under iterative decoding (see [1] for a recent overview of asymptotic results), there is a lack of both code constructions and low-complexity decoding schemes for practical communication scenarios.

It is well-known that maximum-likelihood (ML) decoding of an $[n, k]$ LDPC code (where n is the code length and k is the number of information symbols) with ν erasures is equivalent to solving a system of linear equations of order ν. Thus, it can be performed by Gaussian elimination with time complexity at most $O(\nu^3)$. Exploiting the sparsity of the parity-check matrix of the codes can lower the complexity to approximately $O(\nu^2)$ (see overview and analysis in [2] and references therein). Practically feasible algorithms with a thorough complexity analysis can be found in [3]. However, ML decoding of long LDPC codes of lengths of a few thousand bits over the binary erasure channel (BEC) is still considered completely impractical. A class of low-complexity decoding algorithms for the BEC, based on the concept of guessing bits when conventional belief-propagation (BP) decoding stops, was proposed in [4], [5]. Applying these algorithms to the decoding of LDPC codes improves the performance of BP decoding, but does not provide near-ML performance. A similar method applied to turbo codes does, however, provide near-ML performance, sometimes at low complexity cost (see [6, Fig. 3]).

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In this paper, we propose a decoding algorithm which provides near-ML decoding of long quasi-cyclic (QC) LDPC block codes. The decoding complexity is polynomial in the window length, but only linear in the code length.

The new algorithm is based on a combination of BP decoding of the QC LDPC code followed by so-called "QC sliding-window" ML (WML) decoding. The latter technique is applied "quasi-cyclically" to a relatively short sliding window, where the decoder performs ML decoding of a zero-tail terminated (ZT) LDPC convolutional code. Notice that unlike sliding-window near-ML decoding of convolutional codes considered in [7], the suggested algorithm working on the parent LDPC convolutional code has significantly lower computational complexity due to the sparsity of the code's parity-check matrix [8]. Also, it preserves almost all advantages of the convolutional structure in the sense of erasure correcting capability.

The proposed algorithm resembles a wrap-around suboptimal decoding of tail-biting (TB) convolutional codes [9], [10]. Decoding of a TB code requires identification of the correct starting state, and thus ML decoding must apply the Viterbi algorithm once for each possible starting state. In contrast, wrap-around decoding applies the Viterbi algorithm once to the *wrapped-around* trellis diagram with all starting state metrics initialized to zero. This decoding approach with typically a few passes over the wrapped-around trellis diagram yields near-ML performance at a complexity of a few times the complexity of the Viterbi algorithm.

In order to estimate the achievable finite-length performance of ML decoding of LDPC codes, we first derive an ensemble-average decoding error probability for a specific ensemble $\mathcal{E}$ (E in [11]). The ensemble $\mathcal{E}$ is wider than Gallager's ensemble [12] but easier to analyze, while codes from $\mathcal{E}$ are almost as good as Gallager's codes in terms of distance properties. As expected, the finite-length frame error rate (FER) performance of LDPC codes is close to known bounds on the achievable error probability for general linear codes (see [13] for an overview of bounds and a collection of recent results).

This paper is organized as follows. In Section II, we introduce notations and code descriptions. Next, in Section III, the new decoding algorithm is presented. The ensemble-average finite-length performance of regular LDPC codes is studied in Section IV, while, in Section V, we present simulation results of the new decoding algorithm for QC LDPC codes. Conclusions and a discussion are given in Section VI.

II. PRELIMINARIES

A binary QC LDPC block code can be considered as a TB parent convolutional code determined by a polynomial parity-check matrix whose entries are monomials or zeros.

A rate $R = b/c$ *parent* LDPC convolutional code can be determined by its polynomial parity-check matrix

$$H(D) = \begin{pmatrix} h_{11}(D) & h_{12}(D) & \dots & h_{1c}(D) \\ h_{21}(D) & h_{22}(D) & \dots & h_{2c}(D) \\ \vdots & \vdots & \ddots & \vdots \\ h_{(c-b)1}(D) & h_{(c-b)2}(D) & \dots & h_{(c-b)c}(D) \end{pmatrix} \quad (1)$$

where D is a formal variable, $h_{ij}(D)$ is either zero or a monomial entry, that is, $h_{ij}(D) \in \{0, D^{w_{ij}}\}$ with w_{ij} being a nonnegative integer, and $\mu = \max_{i,j}\{w_{ij}\}$ is the syndrome memory.

The polynomial matrix (1) determines an $[Mc, Mb]$ QC LDPC block code $\mathcal{C}$ using a set of polynomials modulo $D^M - 1$, M being a positive integer. If $M \to \infty$, then we obtain an LDPC convolutional code which is considered as a parent convolutional code with respect to the QC LDPC block code for any finite M. By TB the parent convolutional code to length $M > \mu$, we obtain the binary parity-check matrix

$$H_{\mathrm{TB}} = \begin{pmatrix} H_0 & H_1 & \dots & H_{\mu-1} & H_\mu & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & H_0 & H_1 & \dots & H_{\mu-1} & H_\mu & \dots & \mathbf{0} \\ \vdots & & \ddots & \vdots & \vdots & \vdots & \ddots & \\ H_\mu & \mathbf{0} & \dots & \mathbf{0} & H_0 & H_1 & \dots & H_{\mu-1} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ H_1 & \dots & H_\mu & \mathbf{0} & \dots & \mathbf{0} & \dots & H_0^{\mathrm{T}} \end{pmatrix}$$

of an *equivalent* (in the sense of column permutation) TB code (all matrices H_i including H_{TB} should have a transpose operator to get the exact TB code [14]), where H_i, $i = 0, 1, \dots, \mu$, are binary $(c-b) \times c$ matrices in the series expansion $H(D) = H_0 + H_1 D + \dots + H_\mu D^\mu$.

If every column and row of $H(D)$ contain J and K nonzero entries, respectively, we call $\mathcal{C}$ a (J, K)-regular QC LDPC code and irregular otherwise.

Notice that by zero-tail termination [14] of (1) at length $W > \mu$, we can obtain a parity-check matrix of a $[Wc, (W-\mu)b]$ ZT QC LDPC code.

III. QUASI-CYCLIC SLIDING-WINDOW DECODING OVER THE BINARY ERASURE CHANNEL

Consider a BEC with erasure probability ϵ. Let H be an $M(c-b) \times Mc$ parity-check matrix of a binary $[n = Mc, k = Mb, d_{\min}]$ QC LDPC block code, where $d_{\min}$ is the minimum Hamming distance of the code. An ML decoder corrects any pattern of ν erasures if $\nu \le d_{\min} - 1$. If $d_{\min} \le \nu \le n-k$, then a unique correct decision can be obtained for some erasure patterns. The number of such correctable patterns depends on the code structure.

Let $\boldsymbol{y} = (y_0, y_1, \dots, y_{n-1})$, $n = Mc$, be a received vector, where $y_i \in \{0, 1, \phi\}$, $i = 0, 1, \dots, n-1$, and the symbol ϕ represents an erasure. We denote by $\boldsymbol{e} = (e_0, e_1, \dots, e_{n-1})$ a binary vector, such that for all $i = 0, 1, \dots, n-1$, $e_i = 1$ if $y_i = \phi$, and $e_i = 0$ if $y_i \in \{0,1\}$. Let $I(\boldsymbol{e})$ be the set of nonzero coordinates of $\boldsymbol{e}$, $|I(\boldsymbol{e})| = \nu(\boldsymbol{e})$, and let $\boldsymbol{z}(\boldsymbol{y}) = (z_0, z_1, \dots, z_{\nu-1})$ be a vector of unknowns located in positions of $I(\boldsymbol{e})$. Consider the system of linear equations

$$\boldsymbol{z}(\boldsymbol{y}) H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{s}(\boldsymbol{e}) \quad (2)$$

where $(\cdot)^{\mathrm{T}}$ denotes the transpose of its matrix argument, $\boldsymbol{s}(\boldsymbol{e}) = \boldsymbol{y}_{I^{c}(\boldsymbol{e})} H_{I^{c}(\boldsymbol{e})}^{\mathrm{T}}$ is computed using the nonerased positions of $\boldsymbol{y}$, $I^{c}(\boldsymbol{e}) = \{0, 1, \dots, n-1\} \setminus I(\boldsymbol{e})$, and $H_{I(\boldsymbol{e})}$ is the submatix of H corresponding to $I(\boldsymbol{e})$. ML decoding over the BEC is reduced to solving (2); its complexity for sparse parity-check matrices is of order ν^2, which is still computationally intractable for LDPC codes of large lengths.

In order to reduce decoding complexity, we apply a sliding-window decoding algorithm which is modified for QC LDPC block codes. This decoder is determined by a binary parity-check matrix H_{W} given by

$$H_{\mathrm{W}} = \begin{pmatrix} H_0 & \dots & H_\mu & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & H_0 & \dots & H_\mu & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & H_0 & \dots & H_\mu & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & H_0 & \dots & H_\mu \end{pmatrix} \quad (3)$$

of size $(W-\mu)(c-b) \times Wc$, where $W \ge 2\mu + 1$ denotes the size of the decoding window. The matrix (3) determines a ZT LDPC parent convolutional code. We start decoding with BP decoding applied to the original QC LDPC block code of length $n = Mc$, and then apply ML decoding to the ZT LDPC parent convolutional code determined by the parity-check matrix (3). It implies solving a system of linear equations

$$\boldsymbol{z}(\boldsymbol{y}_i^{i+Wc-1}) H_{I(\boldsymbol{e}),\mathrm{W}}^{\mathrm{T}} = \boldsymbol{s}_{\mathrm{W}}(\boldsymbol{e})$$

where $\boldsymbol{y}_i^{i+Wc-1} = (y_i, y_{i+1 \bmod n}, \dots, y_{i+Wc-1 \bmod n})$, $i = 0, s, 2s, \dots \bmod n$, is a subvector of $\boldsymbol{y}$ corresponding to the chosen window, s (assumed to be a divisor of n) denotes the size of the window shift, and $\boldsymbol{s}_{\mathrm{W}}(\boldsymbol{e})$ and $H_{I(\boldsymbol{e}),\mathrm{W}}$ are the corresponding subvector of $\boldsymbol{s}(\boldsymbol{e})$ and submatrix of $H_{I(\boldsymbol{e})}$, respectively. The final decision is made after $\alpha n/s$ steps, where α denotes the number of passes of sliding-window decoding. The formal description of the decoding procedure is given below as Algorithms 1 and 2.

Notice that the choice of s affects both the performance and the complexity. By increasing s we can speed up the decoding procedure at the cost of some performance loss. In the sequel, we use $s = c$ bits that corresponds to the lowest possible FER.

Algorithm 1 BP-BEC

while there exist parity checks with only one erased symbol **do**

Assign to the erased symbol the modulo-2 sum of all nonerased symbols participating in the same parity check.

end while

IV. ENSEMBLE-AVERAGE DECODING ERROR PROBABILITY

In this section, we derive the ensemble-average ML decoding block error probability $\mathrm{E}_{\mathcal{E}}\{P_{\mathrm{err}}(\mathcal{E}, \epsilon)\}$, where $\mathrm{E}_{\mathcal{E}}\{\cdot\}$

Algorithm 2 Wrap-around algorithm for near-ML decoding of QC LDPC codes over the BEC

Input: $\boldsymbol{y} = (y_0, \ldots, y_{n-1}) \in \{0, 1, \phi\}^n$.
Perform BP decoding on $\boldsymbol{y}$ (using Algorithm 1);
wstart $\leftarrow 0$; wend $\leftarrow Wc - 1$; corrected $\leftarrow 1$;
while corrected > 0 **do**
  corrected $\leftarrow 0$;
  Apply ML decoding to the window $(y_{\text{wstart}}, \ldots, y_{\text{wend}})$;
  wstart $\leftarrow$ wstart $+ s \bmod n$;
  wend $\leftarrow$ wend $+ s \bmod n$;
  if wstart $= 0$ **then**
    corrected $\leftarrow$ number of corrected erasures after a full round;
  end if
end while
return $\boldsymbol{y}$

denotes the ensemble average of its argument, over the BEC with erasure probability ϵ for the ensemble $\mathcal{E}(n, r, K)$ (the ensemble E in [11]) of LDPC codes determined by parity-check matrices chosen with uniform probability from the ensemble of binary $r \times n$ matrices whose row weights are equal to K. This average decoding error probability can be interpreted as an upper bound on the achievable error probability for regular LDPC codes.

For any ensemble $\mathcal{G}(n, r)$ of random $[n, n-r]$ binary linear codes [13]

$$\begin{aligned} \mathrm{E}_{\mathcal{G}}\{P_{\text{err}}(\mathcal{G}, \epsilon)\} &= \sum_{\nu=r+1}^{n} \binom{n}{\nu} \epsilon^{\nu}(1-\epsilon)^{n-\nu} \\ &+ \sum_{\nu=1}^{r} \binom{n}{\nu} \epsilon^{\nu}(1-\epsilon)^{n-\nu} \mathrm{E}_{\mathcal{G}}\{P_{\text{err}}(\mathcal{G}, \epsilon|\nu)\} \end{aligned} \tag{4}$$

where [15]

$$\mathrm{E}_{\mathcal{G}}\{P_{\text{err}}(\mathcal{G}, \epsilon|\nu)\} \le 2^{\nu-r}. \tag{5}$$

Theorem 1. *For the ensemble $\mathcal{E}$ of row-regular LDPC codes with row weight K*

$$\mathrm{E}_{\mathcal{E}}\{P_{\text{err}}(\mathcal{E}, \epsilon|\nu)\} = 2^{\nu-r}\left(1 + \frac{\binom{n-\nu}{K}}{\binom{n}{K}}\right)^{r} - 1. \tag{6}$$

Proof: Our goal is to determine the average error probability $\mathrm{E}_{\mathcal{E}}\{P_{\text{err}}(\mathcal{E}, \epsilon|\nu)\}$ for the ensemble of sparse parity-check matrices described above. This probability is equal to the probability that a solution of (2) is not unique, i.e.,

$$P_{\text{err}}(\mathcal{E}, \epsilon|\nu) = \sum_{\boldsymbol{z}_i, \boldsymbol{z}_j \ne \boldsymbol{z}_i} \Pr\left(\boldsymbol{z}_i H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{z}_j H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{s}(\boldsymbol{e})|\nu\right)$$

where $\Pr\left(\boldsymbol{z}_i H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{z}_j H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{s}(\boldsymbol{e})|\nu\right)$ is the conditional probability that two different subvectors $\boldsymbol{z}_i$ and $\boldsymbol{z}_j$ are solutions of (2), or equivalently

$$\begin{aligned} P_{\text{err}}(\mathcal{E}, \epsilon|\nu) &= \sum_{\boldsymbol{z} \ne \boldsymbol{0}} \Pr\left(\boldsymbol{z} H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{0}|\nu\right) \\ &= \sum_{\boldsymbol{z}} \Pr\left(\boldsymbol{z} H_{I(\boldsymbol{e})}^{\mathrm{T}} = \boldsymbol{0}|\nu\right) - 1. \end{aligned} \tag{7}$$

Interpreting solutions $\boldsymbol{z}$ as equiprobable random vectors with probability $p(\boldsymbol{z}) = 1/2^{\nu}$, (7) can be rewritten as

$$P_{\text{err}}(\mathcal{E}, \epsilon|\nu) = 2^{\nu}\left(\mathrm{E}_{\boldsymbol{z}}\{\Pr\left(\boldsymbol{s}(\boldsymbol{e}) = \boldsymbol{0}|\nu\right)\} - \frac{1}{2^{\nu}}\right).$$

Then,

$$\begin{aligned} \mathrm{E}_{\mathcal{E}}\{P_{\text{err}}(\mathcal{E}, \epsilon|\nu)\} &= 2^{\nu}\left(\mathrm{E}_{\mathcal{E}}\{\mathrm{E}_{\boldsymbol{z}}\{\Pr\left(\boldsymbol{s}(\boldsymbol{e}) = \boldsymbol{0}|\nu\right)\}\} - \frac{1}{2^{\nu}}\right) \\ &= 2^{\nu}\left(\prod_{i=0}^{r-1} \mathrm{E}_{\mathcal{E}}\{\mathrm{E}_{\boldsymbol{z}}\{\Pr(s_i = 0|\nu)\}\} - \frac{1}{2^{\nu}}\right) \end{aligned} \tag{8}$$

where the average probability of a zero syndrome component given ν erasures can be represented as

$$\begin{aligned} \mathrm{E}_{\boldsymbol{z}}\{\Pr(s_i = 0|\nu)\} &= \mathrm{E}_{\boldsymbol{z}}\{\Pr(s_i = 0|w_i = 0, \nu)p(w_i = 0|\nu) \\ &+ \Pr(s_i = 0|w_i \ne 0, \nu)p(w_i \ne 0|\nu)\} \end{aligned}$$

where $p(w_i = 0|\nu)$ denotes the probability of an all-zero row in $H_{I(\boldsymbol{e})}$. Taking into account that $\boldsymbol{z}$ is a random vector with a uniform distribution we can conclude that its product by any nonzero column of $H_{I(\boldsymbol{e})}^{\mathrm{T}}$ is equal to zero with probability $1/2$. Then, we obtain

$$\mathrm{E}_{\boldsymbol{z}}\{\Pr(s_i = 0|\nu)\} = \frac{1 + p(w_i = 0|\nu)}{2}. \tag{9}$$

The ensemble-average probability of an all-zero row in $H_{I(\boldsymbol{e})}$ can be estimated as follows

$$\mathrm{E}_{\mathcal{E}}\{p(w_i = 0|\nu)\} = \frac{\binom{n-\nu}{K}}{\binom{n}{K}} \le \left(\frac{n-\nu}{n}\right)^{K}. \tag{10}$$

Inserting equality from (10) into (9) and (8) yields (6). ■

From the inequality in (10) we can also obtain the simple upper bound

$$\mathrm{E}_{\mathcal{E}}\{P_{\text{err}}(\mathcal{E}, \epsilon|\nu)\} \le 2^{\nu-r}\left(1 + \left(1 - \frac{\nu}{n}\right)^{K}\right)^{r} - 1.$$

A generalization of the bound in (4)–(5) to an ensemble of LDPC codes over the q-ary BEC is presented in [16]. Notice that the upper bound in [16] for $q = 2$ is derived for an LDPC code ensemble with column and row weights growing with the code length (see [16, Thm. 1]).

We do not present here any lower bounds on the ensemble-average error probability in (5), since for rate $R = 1/2$ and code length above 1000, the known lower bound is very close to the upper bound in (5) [13].

V. Numerical Results

We consider three classes of QC LDPC codes parametrized by the integer M (see Section II) and determined by three monomial parity-check matrices of their parent convolutional LDPC codes. Class A contains irregular rate $R = 12/24$ QC LDPC codes of length $1200 - 12000$ ($M \in [50, 500]$) determined by the parity-check matrix

$$H(D) = \begin{pmatrix} H_{\text{bd}}(D) & H_{\text{a}}(D) \end{pmatrix}$$

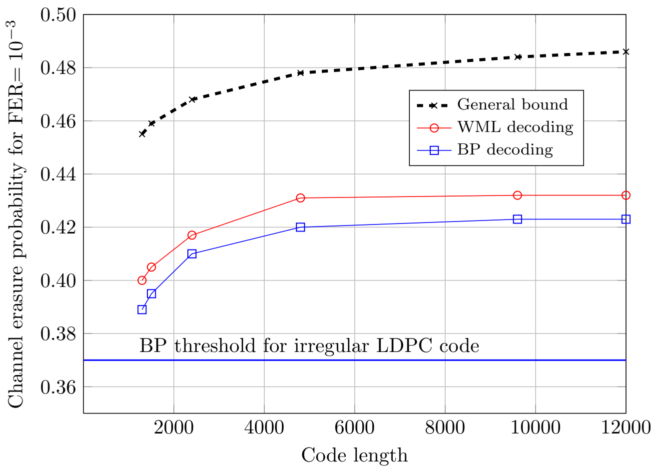

Fig. 1. Channel erasure probability achievable at a FER of 10^{-3} for codes from class A (irregular QC LDPC codes) as a function of code length. Window size is $Wc = 1224$ bits. The blue horizontal line indicates the BP decoding threshold.

where $H_{\rm bd}$ is a bidiagonal matrix of size 12×11 with ones on the diagonals and zeros elsewhere, and $H_{\rm a}$ is a 12×13 matrix whose *degree matrix* is

$$\begin{pmatrix} 0 & -1 & -1 & -1 & 0 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & 16 & 0 & 5 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 12 & 1 & -1 & 7 \\ -1 & -1 & -1 & -1 & 6 & -1 & -1 & 0 & -1 & 15 & 7 & 11 & -1 \\ 13 & -1 & -1 & -1 & -1 & -1 & -1 & 11 & 22 & -1 & 1 & 21 & 4 \\ -1 & -1 & 8 & -1 & -1 & -1 & 21 & -1 & -1 & 12 & 14 & -1 & 19 \\ -1 & -1 & -1 & 0 & -1 & 10 & -1 & -1 & 15 & 21 & -1 & 12 & 3 \\ -1 & 10 & -1 & -1 & -1 & -1 & -1 & 18 & -1 & 5 & 14 & 21 & 23 \\ -1 & -1 & -1 & 3 & -1 & -1 & 8 & -1 & -1 & 18 & 23 & 16 & 11 \\ -1 & -1 & 14 & -1 & 23 & -1 & -1 & -1 & 20 & -1 & 11 & 22 & 7 \\ 0 & -1 & -1 & -1 & -1 & 19 & -1 & -1 & -1 & 18 & 6 & 5 & 22 \\ -1 & 20 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 5 & 11 & 23 & 19 \end{pmatrix}.$$

The polynomial parity-check matrix $H(D)$ is obtained from this degree matrix by replacing each negative entry with a zero, and replacing each nonnegative entry e with D^e. This parity-check matrix was found using the optimization technique described in [17].

Class B contains $(3,6)$-regular rate $R = 3/6$ QC LDPC codes of length $1026 - 12000$ ($M \in [171, 2000]$) determined by the parity-check matrix of [18, Table IV].

Class C contains $(4,8)$-regular rate $R = 8/16$ double-Hamming QC LDPC codes [19] of length $1296 - 12000$ ($M \in [81, 750]$) determined by the degree matrix

$$\begin{pmatrix} 11 & -1 & 4 & -1 & -1 & -1 & -1 & 5 & 6 & 5 & -1 & -1 & 15 & -1 & 3 & 11 \\ 13 & -1 & -1 & 11 & -1 & -1 & 1 & -1 & 13 & 2 & -1 & 2 & -1 & 13 & -1 & 7 \\ 3 & -1 & -1 & -1 & 5 & -1 & 10 & 9 & -1 & 3 & 8 & -1 & -1 & 10 & 8 & -1 \\ 12 & -1 & -1 & -1 & -1 & 1 & 8 & 15 & 9 & -1 & 8 & 4 & 3 & -1 & -1 & -1 \\ -1 & 13 & -1 & 1 & 10 & 12 & 2 & -1 & -1 & -1 & -1 & 0 & -1 & 6 & 6 & -1 \\ -1 & 11 & 6 & -1 & 2 & 6 & -1 & 14 & -1 & -1 & 14 & -1 & 0 & -1 & 2 & -1 \\ -1 & 13 & 3 & 5 & -1 & 5 & -1 & -1 & 10 & -1 & -1 & -1 & 10 & 12 & -1 & 7 \\ -1 & 3 & 4 & 10 & 4 & -1 & -1 & -1 & -1 & 3 & 14 & 11 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

We have used the algorithm in [20], [21] to compute the initial codeword weight and stopping set size spectra of the codes for one particular value of M for each class. The results are tabulated in Table I, where $s_{\min}$ denotes the stopping distance.

In Figs. 1, 2, and 3, the channel erasure probability ϵ achievable at a FER of 10^{-3} by applying standard BP decoding (blue curves) and the new decoding algorithm (red

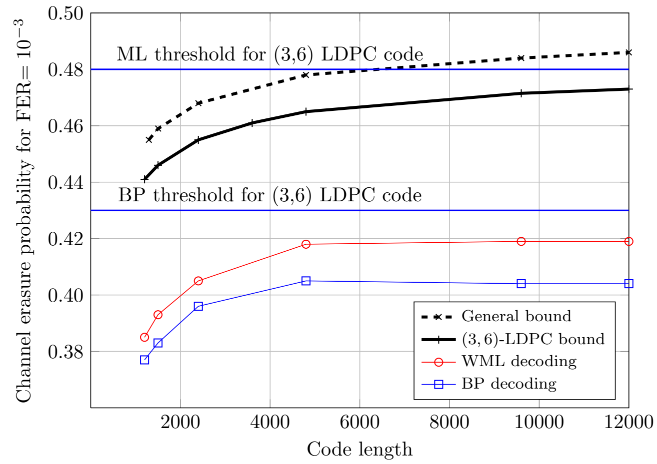

Fig. 2. Channel erasure probability achievable at a FER of 10^{-3} for codes from class B ($(3,6)$-regular QC LDPC codes) as a function of code length. Window size is $Wc = 1026$ bits. The blue horizontal lines indicate the ML and BP decoding thresholds.

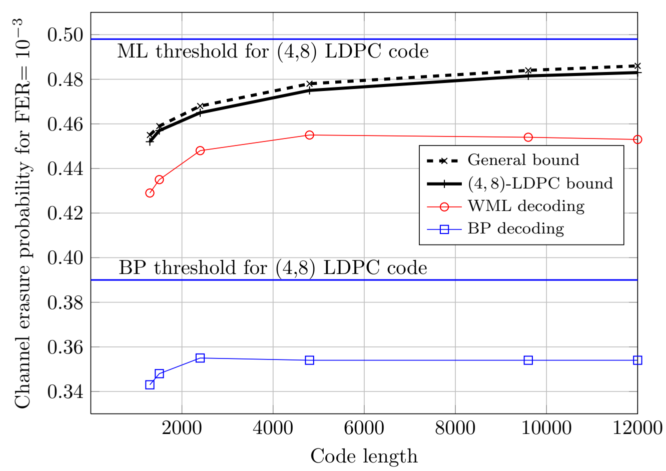

Fig. 3. Channel erasure probability achievable at a FER of 10^{-3} for codes from class C ($(4,8)$-regular QC LDPC codes) as a function of code length. Window size is $Wc = 1296$ bits. The blue horizontal lines indicate the ML and BP decoding thresholds.

curves) to QC LDPC codes from code classes A, B, and C, respectively, as a function of code length, is presented. For comparison, the upper bounds on the ensemble-average decoding error probability for random linear codes (black dashed curve) and for random regular LDPC codes (ensemble $\mathcal{E}$; black solid curve) are presented in the same figures. BP decoding thresholds, computed using density evolution [22], as well as ML decoding thresholds from [23] (blue horizontal lines) are plotted in the same figures for comparison.

In Fig. 4, we present the FER performance of BP and near-ML decoding of three QC LDPC codes of the same length 4800 (one code from each of the code classes A, B, and C) as a function of the channel erasure probability. For comparison, in the same figure, the bit error rate (BER) performance of WML decoding with window size $Wc = 792$ bits of the length-1200 and rate-$12/24$ QC LDPC code from class A, and the performance of contraction-based message-passing (CMP) decoding of two random codes of length 1000 from [4], are presented. Although the window size of the WML decoder is

TABLE I

THE INITIAL CODEWORD WEIGHT AND STOPPING SET SIZE SPECTRA OF CODES FROM THE CODE CLASSES A, B, AND C.

Code class	M	$d_{\min}$	Codeword weight spectrum	$s_{\min}$	Stopping set size spectrum
A [17]	51	18	$(51, 0, 51, 0, 102, 102, 255)$	18	$(51, 51, 51, 51, 153, 153, 510)$
B [18]	171	24	$(2565, 0, 0)$	24	$(2565, 0, 0)$
C [19]	81	> 26		26	(81)

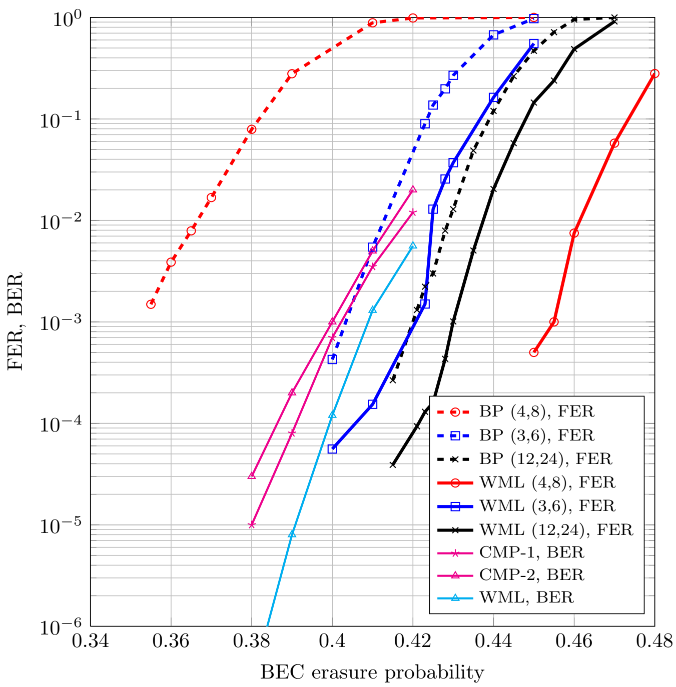

Fig. 4. Comparison of the FER performance of BP and near-ML decoding of QC LDPC codes of length 4800 (one code from each of the three classes).

close to the code length of the CMP decoder, the difference in performance is noticeable.

VI. Discussion and Conclusion

The largest gain of WML decoding with respect to BP decoding is achieved for the $(4, 8)$-regular LDPC codes from class C for which both the difference between ML and BP decoding thresholds and the minimum distance are the largest among all three code classes.

Surprisingly, unlike for regular codes, the BP decoding threshold for irregular codes computed by density evolution is lower than the simulated achievable performance (see Fig. 1). Moreover, WML decoding performance for class A codes is better than that of the $(3, 6)$-regular codes, despite that the minimum distance for the class B codes is larger (see Table I).

For the considered QC LDPC codes, WML decoding outperforms BP decoding. However, due to memory restrictions, WML decoding does not achieve the performance of ML decoding of ensemble-average LDPC codes of the same length, especially if the code length is much larger than the decoding window size.

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