Asymptotic Goodness of Expander Codes with Weak Constituent Codes

Vitaly Skachek

This work is a part of the speaker's Ph.D. Thesis.
It was done at the Technion – Israel Institute of Technology under the supervision of Ron M. Roth.

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- [Richardson Urbanke '01] Good *average* behavior over binary memoryless channels.
- [Richardson Shokrollahi Urbanke '01] Codes, which are extremely close to the capacity, found by the exhaustive search.

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- [Barg Zémor '01–'04] Capacity-achieving codes for BSC with linear-time decoding, exponentially small decoding error. Binary codes that surpass the Zyablov bound.

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- *The relative minimum distance* of $\mathcal{C}$ is defined as $\delta = d/n$.

Linear Code

Definition

- A code $\mathcal{C}$ over field Φ is said to be a *linear* $[n, k, d]$ *code* if there exists a matrix H with n columns and rank $n - k$ such that
$$H\boldsymbol{x}^t = \bar{\mathbf{0}} \;\Leftrightarrow\; \boldsymbol{x} \in \mathcal{C}.$$
- The matrix H is called a *parity-check matrix*.
- The value k is called the *dimension* of the code $\mathcal{C}$.
- The ratio $r = k/n$ is called the *rate* of the code $\mathcal{C}$.
- The words of $\mathcal{C}$ can be obtained as linear combinations of rows of a *generating* $k \times n$ *matrix* G.

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- Linear $[\Delta, k{=}r\Delta, d{=}\delta\Delta]$ code $\mathcal{C}$ over $\mathrm{GF}(q)$.

$\mathbb{C} = (\mathcal{G}, \mathcal{C})$ is the following linear $[N, K, D]$ code over $\mathrm{GF}(q)$:

$$\mathbb{C} = \left\{ \boldsymbol{c} \in (\mathrm{GF}(q))^N \; : \; (\boldsymbol{c})_{E(v)} \in \mathcal{C} \text{ for every } v \in V \right\} \; ,$$

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- The code $\mathbb{C} = (\mathcal{G}, \mathcal{C})$ is a *low-complexity* code.

Low-Complexity Codes – Example

Take $\Delta = 3$, $k = 2$, $|V| = 4$.
Let G be a generating matrix of $\mathcal{C}$
over $F = \mathrm{GF}(2^2) = \{0, 1, \alpha, \alpha^2\}$:

$$G = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & \alpha \end{pmatrix} .$$

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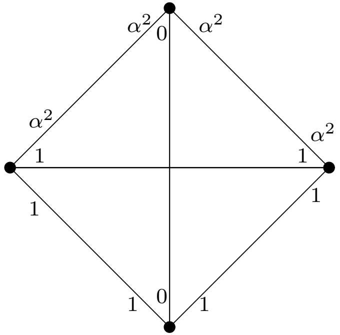

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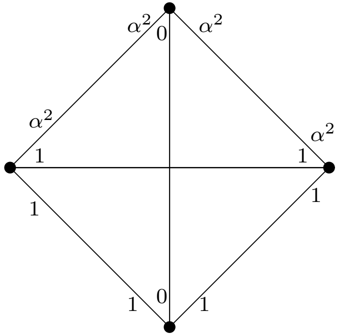

The resulting code $\mathbb{C}$ is of length $N = 6$. For instance,

$$(1\ 1\ \alpha^2\ 0\ \alpha^2\ 1) \in \mathbb{C}.$$

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- The graph $\mathcal{G}$ is an (α, ζ)-*expander* if every subset of at most $\alpha|V|$ vertices expands by a factor of ζ.

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- Let $\lambda_{\mathcal{G}}$ be the second largest eigenvalue of $A_{\mathcal{G}}$.
- Lower ratios of $\gamma_{\mathcal{G}} = \frac{\lambda_{\mathcal{G}}}{\Delta}$ correspond to greater values ζ of expansion. [Alon '86]

Eigenvalues of Expander Graph (cont.)

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- Constructions are due to [Lubotsky Philips Sarnak '88], [Margulis '88].
- For Ramanujan graphs, $\zeta \approx \frac{1}{2}$. Eigenvalue approach cannot provide better bounds on ζ [Kahale '95].

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- For these graphs, $\gamma_{\mathcal{G}} = O(1/\Delta^{1/3})$.

Expander Code Construction

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  - Vertex set $V = A \cup B$ such that $A \cap B = \emptyset$ and $|A| = |B| = n$.
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- Linear $[\Delta, k{=}r_A\Delta, \delta_A\Delta]$ and $[\Delta, r_B\Delta, \delta_B\Delta]$ codes $\mathcal{C}_A$ and $\mathcal{C}_B$, respectively, over $F = \mathrm{GF}(q)$.

Expander Code Construction (cont.)

$\mathbb{C}$ is a linear code of length $|E|$ over F:

$$\mathbb{C} = \left\{ \boldsymbol{c} \in F^{|E|} \; : \; \begin{array}{c} (\boldsymbol{c})_{E(u)} \in \mathcal{C}_A \text{ for every } u \in A \text{ and} \\ (\boldsymbol{c})_{E(u)} \in \mathcal{C}_B \text{ for every } u \in B \end{array} \right\} ,$$

where $(\boldsymbol{c})_{E(u)}$ = the sub-word of $\boldsymbol{c}$ that is indexed by the set of edges incident with u.

Example

Take $k = 2$, $\Delta = 3$, $n = 4$.
Let G_A and G_B be generating matrices of $\mathcal{C}_A$ and $\mathcal{C}_B$ (respectively) over $F = \mathrm{GF}(2^2) = \{0, 1, \alpha, \alpha^2\}$:

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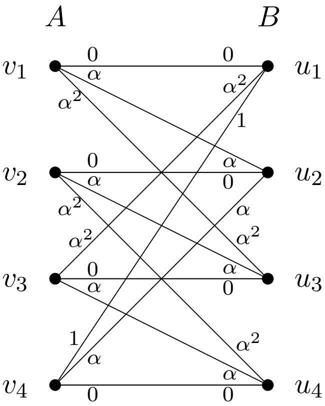

Parameters of Expander Codes

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Relative Minimum Distance

[Roth Skachek '04]

$$D \geq N \cdot \frac{\delta_A \delta_B - \gamma_{\mathcal{G}} \sqrt{\delta_A \delta_B}}{1 - \gamma_{\mathcal{G}}} .$$

Asymptotic Goodness

Definition

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Problem Statement

How weak the constituent codes $\mathcal{C}_A$ and $\mathcal{C}_B$ could be such that the overall expander code will be asymptotically good?

Asymptotic Goodness – Some Answers

The bound on the minimum distance:

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- $\Rightarrow$ Relative minimum distance is bounded away from zero with probability close to 1.

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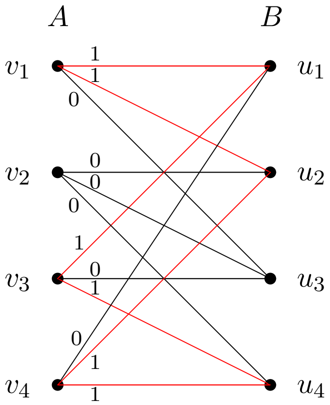

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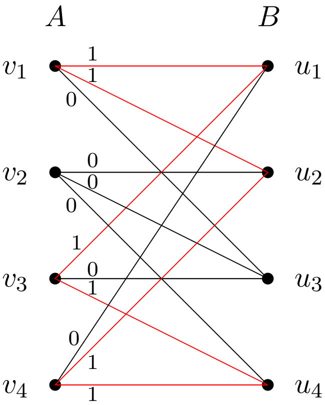

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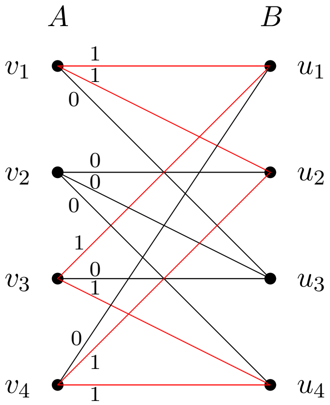


- Every non-zero pattern contains a cycle.
- Every cycle can be converted into a legal non-zero pattern.

Constituent Codes of Minimum Distance 2 (cont.)

Theorem

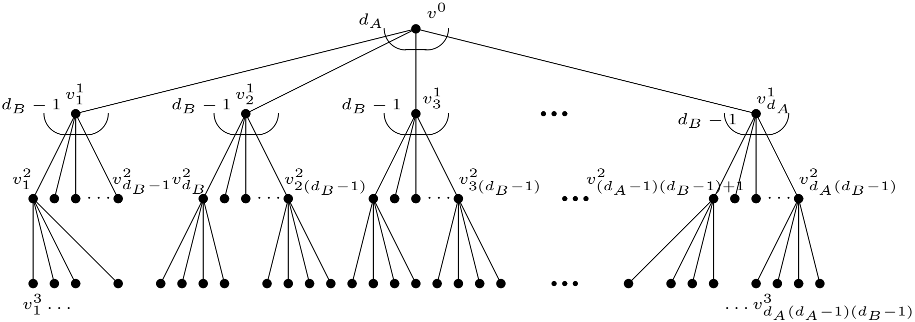
Let $\mathcal{C}_A$ and $\mathcal{C}_B$ be codes of minimum distance 2, and let $\mathcal{G}$ be any Δ-regular bipartite graph. Then, the minimum distance of such code $\mathbb{C}$ is bounded from above by

$$D \leq O\left(\log_{\Delta-1}(n)\right) .$$

Moreover, if the underlying graph $\mathcal{G}$ is a Ramanujan graph as in [Lubotsky Philips Sarnak '88] or [Margulis '88], then the minimum distance of $\mathbb{C}$ is bounded from below by

$$D \geq \frac{4}{3}\log_{\Delta-1}(2n) .$$

Tree-Based Lower Bound



Tree-Based Lower Bound (cont.)

Theorem

Consider the code $\mathbb{C}$ with the constituent codes $\mathcal{C}_A$ and $\mathcal{C}_B$ of minimum distance $d_A \geq 2$ and $d_B \geq 2$, respectively, with the underlying graph $\mathcal{G}$ as in [Lubotsky Philips Sarnak '88] or [Margulis '88]. Then, its relative minimum distance is bounded from below by

$$D \geq (2n)^{1/3 \cdot \log_{\Delta-1}(d_A-1)(d_B-1)} - 1 \, .$$

Sufficient Condition

Theorem

Let $\mathcal{C}_A$ *and* $\mathcal{C}_B(u)$ *(for every* $u \in B$*) be linear codes with the minimum distance* $d_A = \delta_A \Delta$ *and* d_B*, respectively. Let* $\mathcal{G}$ *be a bipartite* (α, ζ)*-expander such that the degree of every* $u \in A$ *is* Δ*. If*

$$\frac{\delta_A}{\zeta + \delta_A - 1} < d_B \ ,$$

then the relative minimum distance of $\mathbb{C}$ *is* $\geq \alpha \delta_A$*.*

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- $\Rightarrow$ Relative minimum distance of $\mathbb{C}$ is $\geq \alpha \delta_A$.
- The previously-known bound does not lead to any interesting result.

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- The new condition improves on the known results for a range of parameters.
- Tight (necessary and sufficient) conditions are still remain an open problem...