On some data processing problems arising in the distributed storage systems

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Joint works with Helger Lipmaa and with Michael Rabbat

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Distributed Storage Systems

- Enormous amounts of data are stored in a huge number of servers.
- Occasionally servers fail.
- Failed server is replaced and the data has to be copied to the new server.

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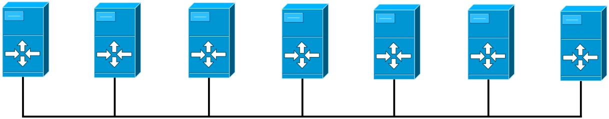

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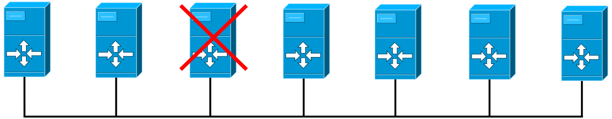

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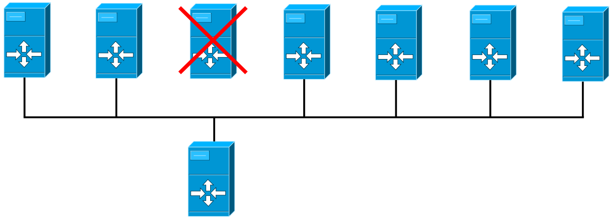

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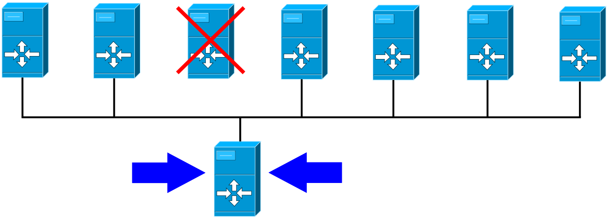

Example: EvenOdd Code

In the context of disk storage: [Blaum, Brady, Bruck, Menon 1995].

Example

$$\begin{array}{c||c||c||c} X_1 & Y_1 & X_1 + Y_1 & X_1 + Y_2 \\ X_2 & Y_2 & X_2 + Y_2 & X_2 + Y_1 \end{array}$$

All the information can be recovered by using any two out of four nodes.

Types of Repair

- Exact repair
- Functional repair
- Exact repair of the systematic part

Functional Repair

- The number of information blocks: M
- The number of information nodes: n
- The total number of active nodes: N
- Number of stored bits per node: α
- Maximal number of nodes used in repair: m
- Number of bits read from each node: t
- Total repair bandwidth: $\gamma = m \cdot t$.

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Here: $N = 4$, $n = 2$, $M = 4$, $m = 2$, $t = 2$ blocks, $\gamma = 4$ blocks.

[Dimakis, Godfrey, Wu, Wainwright, Ramchandran 2008]

Theorem

The following point is feasible:

$$\alpha \geq \begin{cases} \frac{M}{n} & \gamma \in [f(0), +\infty) \\ \frac{M-g(i)\gamma}{n-i} & \gamma \in [f(i), f(i-1)) \end{cases},$$

where

$$f(i) \triangleq \frac{2Mm}{(2n-i-1)i + 2n(m-n+1)}$$

$$g(i) \triangleq \frac{(2m-2n+i+1)i}{2m},$$

and $m < N - 1$.

Special Cases

- **MSR:** Minimum storage regenerating codes.
- **MBR:** Minimum bandwidth regenerating codes.

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MSR codes

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MBR codes

$$(\alpha, \gamma) = \left(\frac{M}{n} \cdot \frac{2N-2}{2N-n-1}, \frac{M}{n} \cdot \frac{2N-2}{2N-n-1} \right) .$$

Code Locality

[Gopalan, Huang, Simitci, Yekhanin 2012]

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Definition

Let $[n, k, d]_q$ be a linear code $\mathcal{C}$ over $\mathbb{F}_q$. We say that the $\mathcal{C}$ has locality r, if the value of any symbol in $\mathcal{C}$ can be recovered by accessing some r other coordinates of $\mathcal{C}$.

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Bound

The following connection holds:

$$n - k \geq \left\lceil \frac{k}{r} \right\rceil + d - 2 \, .$$

The Pyramid codes are shown to achieve this bound.

Code Availability

[Rawat, Papailiopoulos, Dimakis, Vishwanath 2014]

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Bound

The following connection holds:

$$n - k \geq \left\lceil \frac{ks}{r} \right\rceil + d - s - 2 \,.$$

There are explicit constructions of codes that achieve this bound for a variety of parameters.

Batch Codes

- Proposed in [Ishai, Kushilevitz, Ostrovsky, Sahai 2004].
- Can be used in:
  - Load balancing.
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  - Distributed storage systems.

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Constructions:

- [Ishai *et al.* 2004]: algebraic, expander graphs, subsets, RM codes, locally-decodable codes

Prior Art

Design-based constructions and bounds:

- [Stinson, Wei, Paterson 2009]
- [Brualdi, Kiernan, Meyer, Schroeder 2010]
- [Bujtas, Tuza 2011]
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Application to distributed storage:

- [Rawat, Papailiopoulos, Dimakis, Vishwanath 2014]
- [Silberstein 2014]

Batch Codes

Definition [Ishai *et al.* 2004]

$\mathcal{C}$ is an $(n, N, m, M, t)_\Sigma$ batch code over Σ if it encodes any string $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \Sigma^n$ into M strings (buckets) of total length N over Σ, namely $\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_M$, such that for each m-tuple (batch) of (not neccessarily distinct) indices $i_1, i_2, \cdots, i_m \in [n]$, the symbols $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ can be retrieved by m users, respectively, by reading $\leq t$ symbols from each bucket, such that x_{i_ℓ} is recovered from the symbols read by the ℓ-th user alone.

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Definition

If $t = 1$, then we use notation $(n, N, m, M)_\Sigma$ for it. Only one symbol is read from each bucket.

Batch Codes (cont.)

Definition

An $(n, N, m, M, t)_q$ batch code is *linear*, if every symbol in every bucket is a linear combination of original symbols.

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In what follows, consider *linear codes* with $t = 1$ and $N = M$: each encoded bucket contains just one symbol in $\mathbb{F}_q$.

Linear Batch Codes: Our Settings

For simplicity we refer to a linear $(n, N = M, m, M)_q$ batch code as $[M, n, m]_q$ batch code.

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- Let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be an information string.
- Let $\mathbf{y} = (y_1, y_2, \cdots, y_M)$ be an encoding of $\mathbf{x}$.
- Each encoded symbol y_i, $i \in [M]$, is written as $y_i = \sum_{j=1}^{n} g_{j,i} x_j$.
- Form the matrix $\mathbf{G}$:

$$\mathbf{G} = \left(g_{j,i} \right)_{j \in [n], i \in [M]} ;$$

the encoding is $\mathbf{y} = \mathbf{x}\mathbf{G}$.

Code Comparison

Locally repairable codes, codes with locality.

0	1	?	0	1	0	1	1	0

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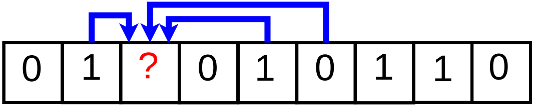
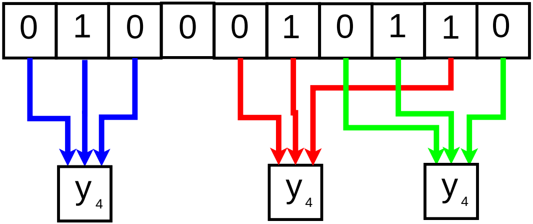
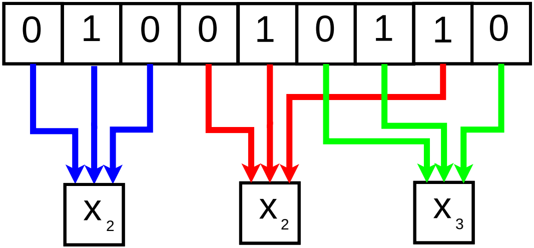


Code Comparison

Codes with locality and availability.



Batch codes.



Theorem

Let $\mathcal{C}$ be an $[M, n, m]_q$ batch code. It is possible to retrieve $x_{i_1}, x_{i_2}, \cdots, x_{i_m}$ simultaneously if and only if there exist m non-intersecting sets $T_1, T_2, \cdots, T_m$ of indices of columns in $\mathbf{G}$, and for T_r there exists a linear combination of columns of $\mathbf{G}$ indexed by that set, which equals to the column vector $\mathbf{e}_{i_r}^T$, for all $r \in [m]$.

Retrieval

Theorem

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Example

[Ishai *et al.* 2004] Consider the following linear binary batch code $\mathcal{C}$ whose 4×9 generator matrix is given by

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} .$$

Retrieval (cont.)

Example

Let $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = \mathbf{x}\mathbf{G}$.
Assume that we want to retrieve the values of (x_1, x_1, x_2, x_2). We can retrieve (x_1, x_1, x_2, x_2) from the following set of equations:

$$\left\{ \begin{array}{lcl} x_1 & = & y_1 \\ x_1 & = & y_2 + y_3 \\ x_2 & = & y_5 + y_8 \\ x_2 & = & y_4 + y_6 + y_7 + y_9 \end{array} \right. .$$

It is straightforward to verify that any 4-tuple $(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4})$, where $i_1, i_2, i_3, i_4 \in [4]$, can be retrieved by using columns indexed by some four non-intersecting sets of indices in [9]. Therefore, the code $\mathcal{C}$ is a $[9, 4, 4]_2$ batch code.

Theorem

Let $\mathcal{C}$ be an $[M, n, m]_2$ batch code $\mathcal{C}$ over $\mathbb{F}_2$. Then, $\mathbf{G}$ *is a generator matrix of the classical error-correcting $[M, n, \geq m]_2$ code.*

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Example

The converse is not true. For example, take $\mathbf{G}$ to be a generator matrix of the classical $[4, 3, 2]_2$ ECC as follows:

$$\mathbf{G} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} .$$

Let $\mathbf{x} = (x_1, x_2, x_3)$. Then, it is impossible to retrieve (x_2, x_3).

Bounds on the Parameters

- Various well-studied properties of linear ECCs, such as MacWilliams identities, apply also to linear batch codes (for $t = 1$, $M = N$ and $q = 2$).

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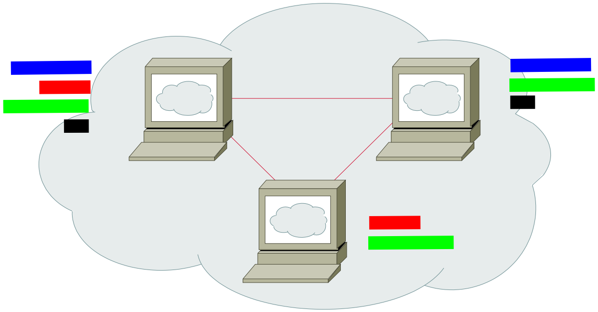
- A variety of bounds on the parameters of ECCs, such as sphere-packing bound, Plotkin bound, Griesmer bound, Elias-Bassalygo bound, McEliece-Rodemich-Rumsey-Welch bound apply to the parameters of $[M, n, m]_2$ batch codes.

File Synchronization Problem



Before synchronization:

- **User** A**:** f_1, f_2, f_3 and f_4.
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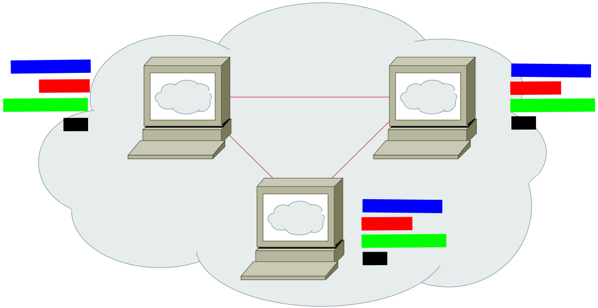

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After synchronization:

- **Users** A, B, C**:** f_1, f_2, f_3 and f_4.

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- Mitzenmacher and Varghese '2012

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Parameters to Consider

- Communication cost $\textsc{Communication}(\mathcal{A})$: the worst case number of bits sent between the devices;
- Computational complexity $\textsc{Computation}(\mathcal{A})$: the worst case number of operations performed at each device;
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- k is the total number of objects in possession of A and B;
- d is the number of objects possessed by only one user;
- u is the size of the space where the objects are taken from.

Prior Art

- Minsky, Trachtenberg and Zippel '2003: characteristic polynomials.
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with high probability.

Subspace Synchronization for Two Users

- Finite field $\mathbb{F}$ with q elements.
- Two users w and v.
- The users own vector spaces $U \subseteq \mathbb{F}^n$ and $V \subseteq \mathbb{F}^n$, respectively.
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Algorithm $\mathcal{A}$

(1) The user w draws a nonzero vector $\mathbf{x} \in U$ randomly and uniformly and communicates it to v.

(2) The node v checks if $\mathbf{x} \in V$. If not, performs

$$V \leftarrow V \oplus \langle \mathbf{x} \rangle \ .$$

(3) Repeat (1)-(2) for $\Theta(d)$ rounds.

(4) Switch the roles of w and v.

Subspace Synchronization for Two Users (cont.)

With high probability,
$\textsc{Communication}(\mathcal{A}) = O(d \cdot n \log q)$,
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The scheme is easily extendable extendable to networks with many users.

Consider a classical $[n, k, d]$-linear code $\mathcal{C}$ over the finite field $\mathbb{F} = \mathbb{F}_q$, such that $n \geq 2^m$ for some integer $m > 0$. (For example, RS code with $n + 1 = k + d$). Let the $(n - k) \times n$ parity-check matrix of $\mathcal{C}$ be

$$H = [\mathbf{h}_1 \mid \mathbf{h}_2 \mid \cdots \mid \mathbf{h}_n] ,$$

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With every vector $\mathbf{x} \in \{0, 1\}^m$ associate a unique integer index $\phi(\mathbf{x}) \in [n]$. If $\mathbf{x}_1 \neq \mathbf{x}_2$, we have $\phi(\mathbf{x}_1) \neq \phi(\mathbf{x}_2)$. Assume that $O = \{\mathbf{x}_i\}_{i \in S}$ is a collection of objects for some $S \subseteq [n]$.

Using Reed-Solomon Codes

Consider a classical $[\mathrm{n},\mathrm{k},\mathrm{d}]$-linear code $\mathcal{C}$ over the finite field $\mathbb{F}=\mathbb{F}_q$, such that $\mathrm{n} \geq 2^m$ for some integer $m>0$. (For example, RS code with $\mathrm{n}+1=\mathrm{k}+\mathrm{d}$). Let the $(\mathrm{n}-\mathrm{k})\times \mathrm{n}$ parity-check matrix of $\mathcal{C}$ be

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Represent the collection O by the vector space

$$\Phi(O) \triangleq \left\langle \mathbf{h}_{\phi(\mathbf{x})} \right\rangle_{\mathbf{x}\in O} .$$

Using Reed-Solomon Codes and Subspaces

In order to perform reconciliation of two sets of objects, O_1 and O_2, the corresponding vector spaces V_1 and V_2 are constructed, such that $V_i = \Phi(O_i)$ for $i = 1, 2$. Then the synchronization algorithm $\mathcal{A}$ is applied to V_1 and V_2.

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Performance

$\textsc{Communication}(\mathcal{A}) = O(\mathrm{d}^2 m) = O(\mathrm{d}^2 \log u)$,
$\textsc{Computation}(\mathcal{A}) = O(\mathrm{d}^2 \cdot u)$.
$\textsc{Time}(\mathcal{A}) = 2$

Network Coding with Hashing Approach

- Denote by $O_A = \{\mathbf{x}_i \in \mathbb{F}^n\}_{i \in \mathcal{X}_A}$ and $O_B = \{\mathbf{x}_i \in \mathbb{F}^n\}_{i \in \mathcal{X}_B}$ the set of objects, which are unique to A and to B, respectively.
- $O_C = \{\mathbf{x}_i \in \mathbb{F}^n\}_{i \in \mathcal{X}_O}$ the set of objects which are possessed by both A and B.
- Let $s = |\mathcal{X}_A|$ and $\tau = |\mathcal{X}_A \cup \mathcal{X}_O|$.
- As before, let $d = |\mathcal{X}_A \cup \mathcal{X}_B|$ be the number of different files for A and B.
- Assume that s, or a tight upper bound on it, is known to both A and B.

User A

- User A creates s arbitrary linear combinations of the form

$$\mathbf{y}_j = \sum_{i \in \mathcal{X}_A \cup \mathcal{X}_O} \alpha_{j,i} \mathbf{x}_i \ , \ j \in [s] \ ,$$

- The protocol uses a hash function $\mathcal{H} : \mathbb{F}^n \to \mathbb{K}$, where $\mathbb{K}$ is the finite set of possible keys.
- User A applies $\mathcal{H}$ to $\mathbf{x}_i$ for all $i \in \mathcal{X}_A \cup \mathcal{X}_O$ to produce hash values $\mathcal{H}(\mathbf{x}_i)$ for all i.
- These values are transmitted to B.

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- User A creates s arbitrary linear combinations of the form

$$\mathbf{y}_j = \sum_{i \in \mathcal{X}_A \cup \mathcal{X}_O} \alpha_{j,i} \mathbf{x}_i \ , \ j \in [s] \ ,$$

- The protocol uses a hash function $\mathcal{H} : \mathbb{F}^n \to \mathbb{K}$, where $\mathbb{K}$ is the finite set of possible keys.
- User A applies $\mathcal{H}$ to $\mathbf{x}_i$ for all $i \in \mathcal{X}_A \cup \mathcal{X}_O$ to produce hash values $\mathcal{H}(\mathbf{x}_i)$ for all i.
- These values are transmitted to B.

A transmits to B the following data:

- the header $\mathbf{h}$, which contains the sorted list of values $\mathcal{H}(\mathbf{x}_i)$, $i \in \mathcal{X}_A \cup \mathcal{X}_O$;
- for all $j \in [s]$, the vector pairs $(\boldsymbol{\alpha}_j, \mathbf{y}_j)$.

Let $\mathbf{X}$ be a $\tau \times n$ matrix over $\mathbb{F}$, whose rows are all vectors $\mathbf{x}_i$ indexed by $[\tau]$. Similarly, let $\mathbf{Y}$ be a $s \times n$ matrix, whose rows are vectors $\mathbf{y}_i$ for all $i \in [s]$. Denote

$$\mathbf{A} = \begin{pmatrix} \alpha_{1,1} & \alpha_{1,2} & \cdots & \alpha_{1,\tau} \\ \alpha_{2,1} & \alpha_{2,2} & \cdots & \alpha_{2,\tau} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{s,1} & \alpha_{s,2} & \cdots & \alpha_{s,\tau} \end{pmatrix} .$$

The transmitted vector pairs can be viewed as the rows of the matrix

$$\mathbf{A} \cdot [\, \mathbf{I}_\tau \mid \mathbf{X} \,] = [\, \mathbf{A} \mid \mathbf{Y} \,] \ ,$$

where

$$\mathbf{X} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_\tau \end{bmatrix} \qquad \text{and} \qquad \mathbf{Y} = \mathbf{A}\mathbf{X} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_s \end{bmatrix} .$$

User B

1. Compute values of the function $\mathcal{H}$ applied to the vectors in its possession. By comparing these values to the values in the header $\mathbf{h}$, it finds the indices corresponding to elements in $\mathcal{X}_O$.
2. For each $j \in [s]$, subtract vectors $\sum_{i \in \mathcal{X}_O} \alpha_{j,i} \mathbf{x}_i$ from $\mathbf{y}_j$. Compute the resulting matrix with s rows:

$$\left[\, \tilde{\mathbf{A}} \mid \tilde{\mathbf{Y}} \, \right] \; ,$$

where rows of $\tilde{\mathbf{Y}}$ are the vectors

$$\tilde{\mathbf{y}}_j = \mathbf{y}_j - \sum_{i \in \mathcal{X}_O} \alpha_{j,i} \mathbf{x}_i \; ,$$

and $\tilde{\mathbf{A}}$ is an invertible $s \times s$ matrix obtained from $\mathbf{A}$ by removing the columns corresponding to the vectors indexed by $\mathcal{X}_O$.

Compute the matrix

$$\left[\ \mathbf{I} \mid \tilde{\mathbf{A}}^{-1}\tilde{\mathbf{Y}}\ \right] = \left[\ \mathbf{I} \mid \tilde{\mathbf{X}}\ \right] ,$$

where, if there are no hashing collisions, $\tilde{\mathbf{X}}$ is exactly the matrix $\mathbf{X}$ having rows corresponding to the vectors indexed by $\mathcal{X}_A$.

Compute the matrix

$$\left[\mathbf{I} \mid \tilde{\mathbf{A}}^{-1} \tilde{\mathbf{Y}} \right] = \left[\mathbf{I} \mid \tilde{\mathbf{X}} \right] ,$$

where, if there are no hashing collisions, $\tilde{\mathbf{X}}$ is exactly the matrix $\mathbf{X}$ having rows corresponding to the vectors indexed by $\mathcal{X}_A$.

Peformance

$\textsc{Communication}(\mathcal{A}) = O(d \cdot n \log q)$
$\textsc{Computation}(\mathcal{A}) = O(k^2 \cdot n)$
If s is known, then $\textsc{Time}(\mathcal{A}) = 2$. If s is not known, then $\textsc{Time}(\mathcal{A}) = 3$.

Using a Pool of Hash Functions

- Large pool of different hash functions (known to both users).
- In each round, the hash function is selected randomly from the pool.
- User A sends to B the ID number of the selected hash function.

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Assume a collection $\mathbb{S}$ of k different files in $\{0,1\}^n$. Let $\mathbb{H}$ be a set of all functions $\mathcal{H} : \{0,1\}^n \to \mathbb{K}$, where $\mathbb{K}$ is the set of all possible keys. Assume that $k \ll |\mathbb{K}| \ll 2^n$.

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Theorem

If $\mathbb{K}$ is selected such that $|\mathbb{K}| > c \cdot (k-1)^2$ for some large constant $c > 0$, then the probability of success is at least $e^{-1/c}$.

Thank you!

Questions?