One-dimensional strong affine representations of the polycyclic monoids

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Let $n \in \mathbb{N}$ be greater than 1. The *polycyclic monoid* $\mathcal{P}_n$ is a monoid with zero given by the presentation

$$\mathcal{P}_n = \langle a_0, \ldots, a_{n-1}, a_0^{-1}, \ldots, a_{n-1}^{-1} : a_i^{-1} a_i = 1 \text{ and } a_i^{-1} a_j = 0, i \neq j \rangle.$$

Let $D = (d_0, d_1, \ldots, d_{n-1})$ be a complete system of residues modulo n and let us consider the functions $f_i \colon \mathbb{Z} \to \mathbb{Z}, x \mapsto nx + d_i$ $(i = 0, 1, \ldots, n-1)$. These functions give rise to a so-called one-dimensional strong affine representation of the polycyclic monoid $\mathcal{P}_n$.

This representation can be visualized by an edge-labeled directed graph: we draw an arrow with label i from x to $f_i(x)$ for each $x \in \mathbb{Z}$ and $i \in \{0, 1, \ldots, n-1\}$. Every connected component of such a graph contains exactly one cycle that can be uniquely identified by the word obtained by recording the labels along the edges of the cycle. It can be shown that the set of words corresponding to the cycles determine the edge-labeled graph up to isomorphism as well as the representation up to equivalence.

Our main problem is describing the set of words corresponding to a one-dimensional strong affine representation of $\mathcal{P}_n$. It has been proven in [1] that every finite set of words can be extended to obtain a set of words describing a representation; however, the exact way of accomplishing that has not been shown. This talk focuses on giving a complete characterization for the sets of words induced by arithmetic sequences starting with zero, i.e., if $D = (0, h, 2h, \ldots, (n-1)h)$ for some positive integer h relatively prime to n. In the case of the bicyclic monoid $\mathcal{P}_2$, this covers all one-dimensional strong affine representations.

References

[1] M. Hartmann, T. Waldhauser, On strong affine representations of the polycyclic monoids, *Semigroup Forum* 97, 2018, no. 1, 87–114.

[1] Joint work with Tamás Waldhauser