Right coherency for monoids

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What's in this talk?

1. What is coherency and why is it interesting?
2. The set-up - S-acts over a monoid S and right congruences
3. Which monoids are coherent?
4. Right coherency for monoids and purity of S-acts
5. Final thoughts

Throughout, S will denote a monoid.

Coherency for monoids: the definition

The monoid S is right coherent

if every finitely generated S-subact of every finitely presented right S-act is finitely presented.

Right coherency is a **finitary condition** i.e. all finite monoids are right coherent.

It is one of a suite of finitary conditions that Nik Ruškuc and I are considering.

Left coherency is defined dually.

1. What is coherency and why is it interesting?
Why is coherency interesting?

- The definition is natural, and fits with that for rings.
- Right coherency is defined in terms of S-acts, and understanding monoids in terms of their acts is surely important.
- It has connections with the *model theory* of S-acts. A monoid is right coherent precisely when the theory of S-acts has a model companion **(Wheeler)**.
- A right coherent monoid has the right ideal Howson property, a.k.a. being finitely right aligned, which has connections via inverse semigroups to C^*-algebras.
- Right coherency has connections with products and ultraproducts of *flat left* S-acts. (**Bulman-Fleming and McDowell, G, Sedaghatjoo**).
- Certain nice classes of monoids are right coherent.
- Right coherency is related to *purity* (weak injectivity) properties of right S-acts.

2. S-acts over a monoid S
Representation of monoid S by mappings of sets

A **right S-act** is a set A together with a map

$$A \times S \to A, \ (a, s) \mapsto as$$

such that for all $a \in A, s, t \in S$

$$a1 = a \text{ and } (as)t = a(st).$$

Beware: an S-act is also called an S-set, S-system, S-action, S-operand, or S-polygon.

Let Act-S denote the class of all right S-acts.

2. S-acts over a monoid S
Standard definitions/Elementary observations

- Right S-acts form a variety of universal algebras to which we may apply the usual notions of subalgebra (**subact**), congruence, morphism, factor/quotient S-act, finitely generated, finitely presented, etc.
- S itself is a right S-act, denoted by S_S.
- Unions and intersections of right S-acts are again right S-acts.
- An S-act A is **finitely generated** (f.g.) if there are $a_1, \cdots, a_n \in A$ with

$$A = a_1 S \cup \cdots \cup a_n S$$

- The **free** S-act $F_S(X)$ on X where $|X| = n$ looks like

$$F_S(X) = x_1 S \dot{\cup} \cdots \dot{\cup} x_n S$$

- A is **finitely presented** (f.p.) if for some f.g. free S-act $F_S(X)$ and f.g. congruence ρ

$$A \cong (x_1 S \cup \cdots \cup x_n S)/\rho.$$

2. S-acts over a monoid S
Right congruences on S

We have observed that S is a right S-act S_S; further, it is free cyclic.

Right congruences

A congruence on S_S is an equivalence relation ρ on S such that for any $s, t, u \in S$ if $s \,\rho\, t$ then $su \,\rho\, tu$.

Such a relation on S is called a **right congruence**.

If ρ is finitely generated, then S/ρ is finitely presented cyclic.

3. Which monoids are right coherent?
First observations

S is right coherent

if every finitely generated S-subact of every finitely presented right S-act is finitely presented.

Theorem: Normak (77)

If S is **right noetherian** (i.e. every right congruence is f.g.) then S is right coherent.

Example: Fountain (92)

There is a monoid S in which every right ideal is f.g. but which is not right coherent.

Let us call the monoid in the example above the **Fountain monoid**: it is constructed from a group and a 4 element nilpotent semigroup.

3. Which monoids are right coherent?
How do we tell?

Theorem: G (1992)

The following are equivalent:

- S is right coherent
- every f.g. subact of S/ρ, where ρ is f.g., is finitely presented;
- a condition involving annihilator right congruences.*

* This is a 'Chase type' condition internal to S that allows us to check for coherency.

3. Which monoids are (are not) right coherent? A selection

Right coherent

- Groups
- Clifford monoids
- Regular weakly right noetherian monoids
- Free commutative monoids
- Free monoids X^*.

Not right coherent

- Free inverse monoids
- Fountain monoid
- $X^* \times X^*$ where $|X| = 3$.

These results are due to many authors: G. (1992); G, Hartmann, Ruškuc (2015); G, Hartmann (2016); G, Hartmann, Ruškuc, Zenab and Yang (2020).

3. Which monoids are right coherent?
Hot off the press

Question: infinite transformation monoids?

For an infinite set X is the full transformation monoid $\mathcal{T}_X$ right or left coherent? Similarly for the symmetric inverse monoid $\mathcal{I}_X$.

Answer: Brookes, G, Ruškuc (2023)

No.

We have shown the negative result by finding a certain configuration of elements g, h, e where e is idempotent and h is a weak inverse of g that will prevent non-right coherency.

4. Coherency and purity
Equations over S-acts

Let A be an S-act. An **equation** over A has the form

$$xs = xt, \; xs = yt \text{ or } xs = a$$

where x, y are variables, $s, t \in S$ and $a \in A$.

Consistency

A set of equations is **consistent** if it has a solution in some S-act $B \supseteq A$.

Algebraic closure

An S-act A is:

- *algebraically closed* if every finite consistent set of equations over A has a solution in A;
- *1-algebraically closed* if every finite consistent set of equations over A in 1 variable has a solution in A.

4. Coherency and purity
Equations over S-acts

We denote by $\mathcal{A}$ and $\mathcal{A}(1)$the classes of algebraically closed and 1-algebraically closed right S-acts, respectively.
The following was inspired by **Wheeler's 76** result for model companions:

G (19[redacted])

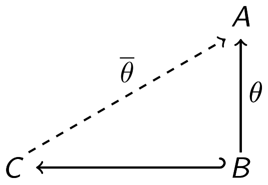
The following are equivalent for a monoid S:

- $\mathcal{A}$ is first order axiomatisable;
- $A(1)$ is first order axiomatisable;
- S is right coherent.

4. Coherency and purity
Injectivity and weak injectivity

Injectivity etc.

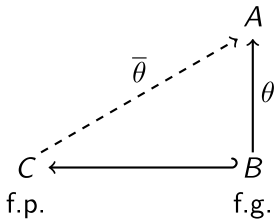
An S-act A is **injective** if for any S-act C and S-subact B, any S-morphism θ from B to A extends to an S-morphism $\overline{\theta}$ from C to A.



4. Coherency and purity
Injectivity and weak injectivity

Injectivity etc.

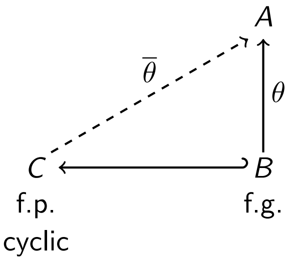
An S-act A is **absolutely pure** if for any S-act C and S-subact B, any S-morphism θ from B to A extends to an S-morphism $\overline{\theta}$ from C to A, where C is finitely presented and B is finitely generated.



4. Coherency and purity
Injectivity and weak injectivity

Injectivity etc.

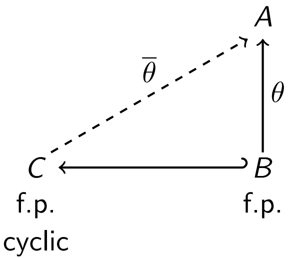
An S-act A is **almost pure** if for any S-act C and S-subact B, any S-morphism θ from B to A extends to an S-morphism $\overline{\theta}$ from C to A, where C is finitely presented cyclic and B is finitely generated.



i.e. $C = S/\rho$ where ρ is a finitely generated right congruence.

Injectivity etc.

An S-act A is **FP injective** if for any S-act C and S-subact B, any S-morphism θ from B to A extends to an S-morphism $\overline{\theta}$ from C to A, where C is finitely presented cyclic and B is finitely presented.



i.e. $C = S/\rho$ where ρ is a finitely generated right congruence.

4. Coherency and purity

- These notions are analogous to those for rings and modules.
- Injectivity of A, and all the weaker notions, are all equivalent to certain consistent sets of equations over A having a solution in A.
- Fact:

$$\begin{aligned} \mathcal{A} &= \text{absolutely pure } S\text{-acts} \\ \mathcal{A}(1) &= \text{almost pure } S\text{-acts} \end{aligned}$$

- **G, Dandan Y (2023)** There is a mechanism to pass between the diagrams and the equations

$$\mathcal{A}^{fp}(1) = \text{fp-injective } S\text{-acts}$$

- Let us call the sets of equations needed above *fp1-equations*.

Injective $\Rightarrow$ absolutely pure/$\mathcal{A}$ $\Rightarrow$ almost pure/$\mathcal{A}(1)$
$\Rightarrow$ FP injective/$\mathcal{A}^{fp}(1)$.

4. Coherency and purity

What does coherency have to do with purity??

Clearly

$$\mathcal{A} \subseteq \mathcal{A}(1) \subseteq \mathcal{A}^{fp}(1).$$

When do these classes coincide?

4. Coherency and purity
How close are and $\mathcal{A}$ and $\mathcal{A}(1)$?

We know from very early work that if $\mathcal{A}(1) = \text{Act-}S$, then $\mathcal{A} = \mathcal{A}(1)$.

We knew from 2016 onwards that if S is finite, then $\mathcal{A} = \mathcal{A}(1)$.

Theorem: Y. Dandan, G (2023)

Let S be a right coherent monoid. Then $\mathcal{A} = \mathcal{A}(1)$.

Question: does $\mathcal{A} = \mathcal{A}(1)$ if and only if S is right coherent?

Answer: No!

Examples: G, Yang, Ruškuc (2023)

The Fountain monoid and $X^* \times X^*$ where $|X| \geq 3$ are examples of non-coherent monoids such that $\mathcal{A} = \mathcal{A}(1)$.

Theorem: B. Lu, Z.K. Liu (2012)

A ring R is right coherent if and only if every FP-injective is absolutely pure.

With *completely* different methods:

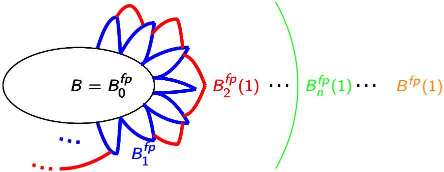
Theorem: G. and Yang (2023)

A monoid S is right coherent if and only if every $\mathcal{A} = \mathcal{A}^{fp}(1)$ if and only if $\mathcal{A}(1) = \mathcal{A}^{fp}(1)$.

4. Coherency and purity
A concrete description

Strategy to show that if every FP-injective is almost pure, i.e. $\mathcal{A}(1) = \mathcal{A}^{fp}(1)$, then S is right coherent

- For an S-act B we build a canonical FP-injective extension $B^{fp}(1)$:

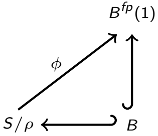


4. Coherency and purity
A concrete description

- Let $B = [b_1]S \cup \cdots \cup [b_k]S$ be a finitely generated subact of S/ρ, where ρ is finitely generated.
- We need to show B has a finite presentation, say via

$$\psi : x_1 S \cup \cdots \cup x_k S \to B, \ x_i \mapsto [b_i].$$

- We have the following diagram, completed due to assumption.



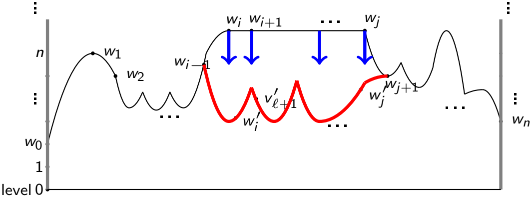
- The morphism ϕ 'fires' [1] into some 'layer' of $B^{fp}(1)$, say to c.

4. Coherency and purity
A concrete description

• By examining the layers we find a finite subset H of $\operatorname{Ker}\psi$.
• If $[b_i]s = [b_j]t$ then

$$cb_is = [1]\phi b_is = [b_i]\phi s = [b_i]s = [b_j]t = \cdots = cb_jt.$$

• This gives us a 'widget' sequence; the widgets have 'levels' depending on where they lie in $B^{fp}(1)$
• We then 'pull' this sequence down through the layers of $B^{fp}(1)$ till all widgets have level 0
• This gives us a finite set of steps using only H to get from x_is to x_jt.



5. Final thoughts
Questions

- Characterise those S such that $\mathcal{A} = \mathcal{A}(1)$.
- Do the same for rings.
- When is $\mathcal{E} = \mathcal{E}(1)$, where $\mathcal{E}$ ($\mathcal{E}(1)$) are the class of (1-) existentially closed S-acts.
- Determine exact connections of right coherency with products/ultraproducts of flat **left** S-acts.
- Other finitary conditions arise from model theoretic considerations of S-acts; many open questions remain!

6. Final thoughts
Selected references

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Thank you for listening

Sincere thanks to the organisers!