ON INJECTIVE HULLS OF S-POSETS

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Abstract. In this paper we describe injectives in the category of S-posets with S-submultiplicative morphisms and construct injective hulls of S-posets with respect to a specific class $\mathcal{E}_{\leqslant}$ of monomorphisms.

1. Introduction

Injectivity is among properties that were studied in the very first articles which appeared in the area of S-posets ([12] and [5]). It is also a topic of several recent papers on S-posets (see, e.g., [3] and [13]). From those articles it turns out that a regularly injective S-poset has to be complete in such a way that suprema are compatible with S-action. One of the aims of this paper is to show that in a suitable framework injectivity is equivalent to that kind of completeness. The second goal is to show how to construct injective hulls of S-posets with respect to a certain class of monomorphisms.

In this work, S is always a *pomonoid*, that is, a monoid S equipped with a partial order $\leqslant$ such that $ss' \leqslant tt'$ whenever $s \leqslant t$, $s' \leqslant t'$ in S. A poset $(A, \leqslant)$ together with a mapping $A \times S \to A$ (under which a pair (a, s) maps to an element of A denoted by as) is called a *right S-poset*, denoted by A_S, if for any $a, b \in A$, $s, t \in S$,

(1) $a(st) = (as)t$,
(2) $a1 = a$,
(3) $a \leqslant b,\ s \leqslant t$ imply that $as \leqslant bt$.

A left S-poset can be defined similarly. *Right S-poset homomorphisms* are order-preserving mappings which also preserve the S-action. We denote the category of right S-posets with S-poset homomorphisms as morphisms by Pos_S. An *S-subposet* of an S-poset A_S is an action-closed subset of A whose partial order is the restriction of the order of A.

Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$. We recall that an object Q from $\mathcal{C}$ is *$\mathcal{M}$-injective* in $\mathcal{C}$ provided that for any morphism $h : A \to B$ in $\mathcal{M}$ and any morphism $f : A \to Q$ in $\mathcal{C}$ there exists a morphism $g : B \to Q$ in $\mathcal{C}$ such that $gh = f$.

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A morphism $\eta : A \to B$ in $\mathcal{M}$ is called $\mathcal{M}$-*essential* if every morphism $\psi : B \to C$ in $\mathcal{C}$, for which the composite $\psi\eta$ is in $\mathcal{M}$, is itself in $\mathcal{M}$. An object $H \in \mathcal{C}$ is called an $\mathcal{M}$-*injective hull* of an object $A \in \mathcal{C}$ if H is $\mathcal{M}$-injective and there exists an $\mathcal{M}$-essential morphism $A \to H$ (see [1], Def. 9.22).

Let A_S and B_S be S-posets. We say that a mapping $f : A \to B$ is S-*submultiplicative* if $f(a)s \leqslant f(as)$ for any $a \in A$, $s \in S$. We denote by $\mathsf{Pos}_S^{\leqslant}$ the category where objects are right S-posets and morphisms are S-submultiplicative order-preserving mappings. Clearly every S-poset homomorphism is an S-submultiplicative order-preserving mapping, so Pos_S is a subcategory of $\mathsf{Pos}_S^{\leqslant}$ which is not necessarily full.

Example 1. Consider the pomonoid $S = (\{0,1\}, \cdot, \leqslant)$ where $0 < 1$. Then the constant mapping

$$f : S_S \to S_S, 1 \mapsto 1, 0 \mapsto 1$$

is order-preserving and S-submultiplicative, but it is not a right S-act homomorphism, because $f(1) \cdot 0 = 0 \neq 1 = f(1 \cdot 0)$.

An *order embedding* from a poset $(A, \leqslant_A)$ to a poset $(B, \leqslant_B)$ is a mapping $h : A \to B$ such that $a \leqslant_A a'$ iff $h(a) \leqslant_B h(a')$, for all $a, a' \in A$. Every order embedding is necessarily an injective mapping. We will denote by $\mathcal{E}$ the class of all right S-poset homomorphisms that are order-embeddings. These are precisely the regular monomorphisms in Pos_S (see [2], Theorem 7).

In this paper we will study injectivity with respect to a specific class of order-embeddings. Let $\mathcal{E}_{\leqslant}$ be the class of morphisms $e : A_S \to B_S$ in the category $\mathsf{Pos}_S^{\leqslant}$ which satisfy the following condition: $e(a)s \leqslant e(a')$ implies $as \leqslant a'$ for all $a, a' \in A$ and $s \in S$. Evidently, each morphism in $\mathcal{E}_{\leqslant}$ is an order-embedding. On the other hand, every S-poset homomorphism that is an order embedding belongs to $\mathcal{E}_{\leqslant}$. In other words, $\mathcal{E} \subseteq \mathcal{E}_{\leqslant}$.

Lemma 2. *Let S be a pogroup. Then $\mathsf{Pos}_S^{\leqslant} = \mathsf{Pos}_S$ and $\mathcal{E}_{\leqslant} = \mathcal{E}$.*

Proof. We need to show that every order-preserving S-submultiplicative mapping $f : A_S \to B_S$ is a right S-poset homomorphism. For every $a \in A$ and $s \in S$ we have $f(a)s \leqslant f(as)$ and $f(as)s^{-1} \leqslant f(ass^{-1}) = f(a)$ by S-submultiplicativity. Multiplying both sides of the last inequality by s we obtain $f(as) \leqslant f(a)s$. We conclude that $f(as) = f(a)s$, as required.

For the second claim we have to prove the inclusion $\mathcal{E}_{\leqslant} \subseteq \mathcal{E}$. If $e \in \mathcal{E}_{\leqslant}$ then clearly e is an order-embedding. By the first part of this proof, e also preserves S-action. □

Inspired by the notion of a quantale (see [8]) we introduce a term "S-quantale".

Definition 3. We call a right S-poset A_S a *right S-quantale* if

(1) the poset A is a complete lattice;
(2) $(\bigvee M)s = \bigvee\{ms \mid m \in M\}$ for each subset M of A and each $s \in S$.

In the following, "S-quantale" is also used to substitute the term "right S-quantale".

We note that S-quantales also appear in [5] under the name of "complete S-posets", in [4] under the name of "continuously complete S-posets" and in [10] under the name of "equivariantly complete S-posets".

Injectivity properties of S-posets have been studied by several authors. First of all, injectivity with respect to all monomorphisms (i.e., injective homomorphisms) is not an interesting property, because the only S-posets with this property are singletons ([3], Theorem 2.5). Therefore it is more natural to study $\mathcal{E}$-injectivity. Since the morphisms in $\mathcal{E}$ are precisely the regular monomorphisms of Pos_S, $\mathcal{E}$-injective S-posets have been called also *regularly injective* (cf. [13]).

The first to study regular injectivity was Skornyakov ([12]) who studied S-posets over a discretely ordered monoid S and proved that such S-posets are complete as posets if they are regularly injective. Later on, Fakhruddin generalized this result to arbitrary pomonoids. He showed that (using our terminology), a regularly injective left S-poset over a pomonoid S is necessarily a left S-quantale (see [5], Proposition 7.2). Recently, Ebrahimi, Mahmoudi and Rasouli showed in their paper [3] that for a pomonoid S, an S-poset is regularly injective if and only if it is a retract of a cofree S-poset over a complete poset. However, it seems that there exist no descriptions of regularly injective S-posets A_S in terms that are internal to A_S. There are some special results. For example, Fakhruddin in [5] found that for a pogroup S, a left S-poset is regularly injective if and only if it is a left S-quantale, thereby generalizing a similar result of Skornyakov.

One approach to obtain necessary and sufficient conditions for injectivity is to allow a larger class of morphisms between S-posets. This is inspired by the most recent work [7], and also [14], in which certain injective hulls of posemigroups were constructed in a category where morphisms are submultiplicative order-preserving mappings. In this work, we will first investigate $\mathcal{E}_{\leqslant}$-injectives in the category $\mathsf{Pos}_S^{\leqslant}$ (which has the same objects but possibly more morphisms than Pos_S) and then give an explicit construction of $\mathcal{E}_{\leqslant}$-injective hulls of S-posets in $\mathsf{Pos}_S^{\leqslant}$.

2. $\mathcal{E}_{\leqslant}$-injective S-posets

In this section we show that $\mathcal{E}_{\leqslant}$-injective objects in the category $\mathsf{Pos}_S^{\leqslant}$ are precisely the right S-quantales.

Proposition 4. *Let Q_S be an S-quantale. Then Q_S is $\mathcal{E}_{\leqslant}$-injective in the category $\mathsf{Pos}_S^{\leqslant}$.*

Proof. Let Q_S be an S-quantale, $e : A_S \to B_S$ be a morphism in $\mathcal{E}_{\leqslant}$ and let $f : A_S \to Q_S$ be a morphism in $\mathsf{Pos}_S^{\leqslant}$. Define a mapping $g : B_S \to Q_S$ by

$$g(b) = \bigvee\{f(a)z \mid e(a)z \leqslant b,\ a \in A,\ z \in S\},$$

for any $b \in B$. Then g is obviously an order-preserving mapping. For any $s \in S$, we have

$$\begin{aligned} g(b)s &= \Big(\bigvee\{f(a)z \mid e(a)z \leqslant b,\ a \in A,\ z \in S\}\Big)\, s \\ &= \bigvee\{f(a)zs \mid e(a)z \leqslant b,\ a \in A,\ z \in S\} \\ &\leqslant \bigvee\{f(a)t \mid e(a)t \leqslant bs,\ a \in A,\ t \in S\} \\ &= g(bs), \end{aligned}$$

which means that g is S-submultiplicative. (Note that here we used that $e(a)z \leqslant b$ implies $e(a)zs \leqslant bs$ if $a \in A,\ z \in S$.) Finally, for any $a \in A$, we have

$$g(e(a)) = \bigvee\{f(x)z \mid e(x)z \leqslant e(a),\ x \in A,\ z \in S\}.$$

If $x \in A$, $z \in S$ are such that $e(x)z \leqslant e(a)$ then $xz \leqslant a$ and hence

$$f(x)z \leqslant f(xz) \leqslant f(a).$$

Consequently, $(ge)(a) \leqslant f(a)$. On the other hand, $f(a)$ is obviously one of the terms in the sup that defines $(ge)(a)$. Therefore, $ge = f$ as needed. □

Proposition 5. *In the category $\mathsf{Pos}_S^{\leqslant}$, every retract of an S-quantale is an S-quantale.*

Proof. Let E_S be an S-quantale and let A_S be a retract of E_S. Then there exist S-submultiplicative order-preserving mappings $i : A \to E$ and $g : E \to A$ such that $gi = id_A$, where id_A is the identity mapping on A. It is obvious that A is complete.

Let $s \in S$, $M \subseteq A$. Clearly, $(\bigvee M)s$ is an upper bound of $\{ms \mid m \in M\}$. Suppose that u is an upper bound of $\{ms \mid m \in M\}$ in A. Then

$$\begin{aligned} u &= g(i(u)) \geqslant g\left(\bigvee_E \{i(ms) \mid m \in M\}\right) \geqslant g\left(\bigvee_E \{i(m)s \mid m \in M\}\right) \\ &= g\left(\left(\bigvee_E \{i(m) \mid m \in M\}\right) s\right) \geqslant g\left(\bigvee_E \{i(m) \mid m \in M\}\right) s \geqslant \left(\bigvee_A M\right) s. \end{aligned}$$

This means that $(\bigvee M)s$ is the least upper bound of $\{ms \mid m \in M\}$, that is,

$$\left(\bigvee M\right) s = \bigvee \{ms \mid m \in M\}.$$

□

A subset D of a poset A is said to be a *down-set* if $x \leqslant d$ implies that $x \in D$ for any $x \in A$, $d \in D$. For any $D \subseteq A$, we denote by $D{\downarrow}$ the down-set $\{x \in A \mid x \leqslant d$ for some $d \in D\}$ and by $a{\downarrow}$ the down-set $\{x \in A \mid x \leqslant a\}$ for $a \in A$.

Now we wish to construct an $\mathcal{E}_{\leqslant}$-injective S-poset starting from an arbitrary right S-poset.

Let A_S be an S-poset, and let $\mathscr{P}(A)$ be the set of all down-sets of the poset A. Define a right S-action $\cdot$ on $\mathscr{P}(A)$ by

$$D \cdot s = (Ds){\downarrow} = \{x \in A \mid x \leqslant ds \text{ for some } d \in D\},$$

for any $s \in S$, $D \in \mathscr{P}(A)$. It is routine to check that $(\mathscr{P}(A), \cdot)$ is an S-act, and an S-poset if we consider inclusion as the partial order. Furthermore, $(\mathscr{P}(A), \cdot, \subseteq)$ is a right S-quantale, that is, $\mathscr{P}(A)$ is a complete lattice under the inclusion relation with supremum being union, and it satisfies

$$\left(\bigvee \{M_\alpha \mid \alpha \in \Omega\}\right) \cdot s = \bigvee \{M_\alpha \cdot s \mid \alpha \in \Omega\}$$

for any $M_\alpha \in \mathscr{P}(A)$, $\alpha \in \Omega$, $s \in S$. We denote the right S-quantale $(\mathscr{P}(A), \cdot, \subseteq)$ shortly by $\mathscr{P}(A)_S$. By Proposition 4, we have the following result.

Proposition 6. *Let A_S be an S-poset. Then $\mathscr{P}(A)_S$ is $\mathcal{E}_{\leqslant}$-injective in the category $\mathsf{Pos}_S^{\leqslant}$.*

Using the above construction, we obtain a description of $\mathcal{E}_{\leqslant}$-injectives in the category $\mathsf{Pos}_S^{\leqslant}$ in the next theorem.

Theorem 7. *Let A_S be an S-poset. Then A_S is $\mathcal{E}_{\leqslant}$-injective in $\mathsf{Pos}_S^{\leqslant}$ if and only if A_S is a right S-quantale.*

Proof. **Necessity.** The mapping $\eta : A_S \to \mathscr{P}(A)_S$ given by $\eta(a) = a{\downarrow}$ for each $a \in S$ is clearly an order embedding of the poset A into the poset $\mathscr{P}(A)$. It is routine to check that η preserves S-action and hence η is also S-submultiplicative. Moreover, if $\eta(a) \cdot s \subseteq \eta(a')$ for $a, a' \in A,\ s \in S$, then $(as){\downarrow} = a{\downarrow} \cdot s \subseteq a'{\downarrow}$. This implies that $as \leqslant a'$, which means that $\eta \in \mathcal{E}_{\leqslant}$.

Since A_S is $\mathcal{E}_{\leqslant}$-injective by assumption, A_S is a retract of the S-quantale $\mathscr{P}(A)_S$. Consequently, A_S is an S-quantale by Proposition 5.

Sufficiency follows by Proposition 4. □

Corollary 8 (Cf. [12] and [5])**.** *For a right S-poset A_S over a pogroup S the following assertions are equivalent.*

(1) *A_S is $\mathcal{E}$-injective in Pos_S.*
(2) *A_S is $\mathcal{E}_{\leqslant}$-injective in $\mathsf{Pos}_S^{\leqslant}$.*
(3) *A_S is a right S-quantale.*

Proof. (1) ⇔ (2) by Lemma 2. (2) ⇔ (3) by Theorem 7. □

We say that a pomonoid S is *right (left) $\mathcal{E}_{\leqslant}$-self-injective* if the S-poset S_S (${}_SS$) is $\mathcal{E}_{\leqslant}$-injective in the category $\mathsf{Pos}_S^{\leqslant}$ (${}_S\mathsf{Pos}^{\leqslant}$). A pomonoid S is *$\mathcal{E}_{\leqslant}$-self-injective* if it is both right and left $\mathcal{E}_{\leqslant}$-self-injective.

Self-injective (unordered) semigroups have been studied by several authors (see the comments in [6]). In particular, Päeva [9] has given necessary and sufficient conditions for right self-injectivity of a semigroup in terms of certain homomorphisms and right congruences. From Theorem 7 it immediately follows that $\mathcal{E}_{\leqslant}$-self-injectivity of a pomonoid can be described in a quite simple way.

Corollary 9. *A pomonoid S is $\mathcal{E}_{\leqslant}$-self-injective if and only if it is a quantale.*

3. $\mathcal{E}_{\leqslant}$-injective hulls of S-posets

In a recent article [7], Lambek et al considered injective hulls in the category of pomonoids and submultiplicative order-preserving mappings. Later on, Zhang and Laan in [14] extended those results to certain posemigroups and submultiplicative order-preserving mappings. Inspired by these results, in this section, we construct $\mathcal{E}_{\leqslant}$-injective hulls in the category $\mathsf{Pos}_S^{\leqslant}$. Similarly to Proposition 2.1 in [7] it can be shown that $\mathcal{E}_{\leqslant}$-injective hulls are unique up to isomorphism.

Recall that an order-preserving mapping j on a poset P is called a *closure operator* if it satisfies

(1) $a \leqslant j(a)$,
(2) $j(j(a)) = j(a)$,

for all $a \in P$. Let us introduce the concept of S-quantic nucleus similarly to the case of quantales (see [11]).

Definition 10. Let Q_S be an S-quantale. We say that an S-submultiplicative closure operator j on Q is an *S-quantic nucleus.*

Lemma 11. *Let j be an S-quantic nucleus on an S-quantale Q_S. Then $j(as) = j(j(a)s)$ for any $a \in Q, s \in S$.*

Proof. On one hand, since j is increasing and order-preserving it follows that

$$a \leqslant j(a) \Rightarrow as \leqslant j(a)s \Rightarrow j(as) \leqslant j(j(a)s).$$

Conversely, $j(a)s \leqslant j(as)$ implies $j(j(a)s) \leqslant j(j(as)) = j(as)$. □

Given a closure operator j on a complete lattice Q, its subset $Q_j = \{a \in Q \mid j(a) = a\}$ is again complete. Moreover, $\bigvee\{j(a) \mid a \in Q\} = j(\bigvee\{a \mid a \in Q\})$ (see [11]).

Theorem 12. *If $j : Q_S \to Q_S$ is an S-quantic nucleus, then Q_j is an S-quantale with the action $a \circ s = j(as)$.*

Proof. Since

$$\begin{aligned}(a \circ s) \circ t &= j((a \circ s)t) = j(j(as)t) = j((as)t) = j(a(st)) = a \circ (st),\\ a \circ 1 &= j(a1) = j(a) = a\end{aligned}$$

for all $a \in Q_j$ and $s, t \in S$, $(Q_j, \circ)$ is an S-act. Also, if $a \leqslant b$ and $s \leqslant t$ then $as \leqslant bt$ and $a \circ s = j(as) \leqslant j(bt) = b \circ t$, so $(Q_j, \circ)$ is an S-poset.

Let us show that $(\bigvee M) \circ s = \bigvee\{m \circ s \mid m \in M\}$ for any $M \subseteq Q, s \in S$. Obviously, $\bigvee\{m \circ s \mid m \in M\} \leq (\bigvee M) \circ s$. Conversely, we have

$$\begin{aligned}\left(\bigvee M\right) \circ s &= j\left(\left(\bigvee M\right) s\right) = j\left(\bigvee\{ms \mid m \in M\}\right)\\ &\leqslant j\left(\bigvee\{j(ms) \mid m \in M\}\right) = \bigvee\{j(j(ms)) \mid m \in M\}\\ &= \bigvee\{j(ms) \mid m \in M\} = \bigvee\{m \circ s \mid m \in M\}.\end{aligned}$$

So $(Q_j, \circ)$ is an S-quantale. □

In the next step, we will construct an $\mathcal{E}_{\leqslant}$-injective hull for any S-poset A_S in the category $\mathsf{Pos}_S^{\leqslant}$.

For any down-set D of an S-poset A_S we define its closure by

$$\mathsf{cl}(D) := \{x \in A \mid Ds \subseteq a{\downarrow} \text{ implies } xs \leqslant a \text{ for all } a \in A,\ s \in S\}.$$

Lemma 13. *For an S-poset A_S, cl is an S-quantic nucleus on $\mathscr{P}(A)_S$.*

Proof. It is straightforward to show that cl is a closure operator on $\mathscr{P}(A)_S$. Let us prove that cl is S-submultiplicative, i.e., $\mathsf{cl}(D) \cdot s \subseteq \mathsf{cl}(D \cdot s)$, for any $D \in \mathscr{P}(A)_S, s \in S$, Take $y \in \mathsf{cl}(D) \cdot s$. Then $y \leqslant xs$ for some $x \in \mathsf{cl}(D)$. Suppose that $(D \cdot s)t \subseteq a{\downarrow}$ where $t \in S$ and $a \in A$. Then for any $d \in D$, $dst \in (D \cdot s)t \subseteq a{\downarrow}$. Hence $D(st) \subseteq a{\downarrow}$. So $x(st) \leqslant a$ because $x \in \mathsf{cl}(D)$. It follows that $yt \leqslant xst \leqslant a$, which results in $y \in \mathsf{cl}(D \cdot s)$, as needed. □

For any S-poset A_S, we put

$$\mathscr{Q}(A) := \mathscr{P}(A)_{\mathsf{cl}} = \{D \in \mathscr{P}(A) \mid \mathsf{cl}(D) = D\}$$

and define a right S-action $\circ$ on $\mathscr{Q}(A)$ by

$$D \circ s := \mathsf{cl}(D \cdot s),$$

for any $s \in S$. By Theorem 12, the S-poset $\mathscr{Q}(A)_S = (\mathscr{Q}(A), \circ, \subseteq)$ is an S-quantale, and hence it is $\mathcal{E}_{\leqslant}$-injective in the category $\mathsf{Pos}_S^{\leqslant}$.

Now we prove our main theorem.

Theorem 14. *For every S-poset A_S, $\mathscr{Q}(A)_S$ is the $\mathcal{E}_{\leqslant}$-injective hull of A_S in the category $\mathsf{Pos}_S^{\leqslant}$.*

Proof. Since S is a monoid, $a\!\downarrow \in \mathscr{Q}(A)_S$ for any $a \in A$. We will show that the mapping $\eta : A_S \to \mathscr{Q}(A)_S, a \mapsto a\!\downarrow$ is an $\mathcal{E}_\leqslant$-essential morphism in $\mathsf{Pos}_S^\leqslant$.

To show that η is an S-poset homomorphism, take $a \in A,\ s \in S$. It is easy to see that $a\!\downarrow \cdot\, s = (as)\!\downarrow$. Hence we have

$$\eta(a) \circ s \quad = \quad \mathsf{cl}(a\!\downarrow \cdot\, s) = \mathsf{cl}((as)\!\downarrow) = (as)\!\downarrow = \eta(as),$$

i.e. η is an S-act homomorphism. For every $a, b \in A$, $a \leqslant b$ if and only if $a\!\downarrow \subseteq b\!\downarrow$, which means that η is an order embedding. Thus $\eta \in \mathcal{E} \subseteq \mathcal{E}_\leqslant$.

Finally, let $\psi : \mathscr{Q}(A)_S \to B_S$ be a morphism in $\mathsf{Pos}_S^\leqslant$ such that $\psi\eta \in \mathcal{E}_\leqslant$. We have to show that $\psi \in \mathcal{E}_\leqslant$. Suppose that $\psi(D)z \leqslant \psi(D')$, where $D, D' \in \mathscr{Q}(A),\ z \in S$. First we prove that

$$(\forall a \in A,\ s \in S)(D's \subseteq a\!\downarrow \Longrightarrow Dzs \subseteq a\!\downarrow). \tag{3.1}$$

Assume that $D's \subseteq a\!\downarrow$, $a \in A$, $s \in S$, and take an element $d \in D$. Then $D' \circ s = \mathsf{cl}((D's)\!\downarrow) \subseteq \mathsf{cl}((a\!\downarrow)\!\downarrow) = \mathsf{cl}(a\!\downarrow) = a\!\downarrow$ and so

$$(\psi\eta)(d)zs = \psi(d\!\downarrow)zs \leqslant \psi(D)zs \leqslant \psi(D')s \leqslant \psi(D' \circ s) \leqslant \psi(a\!\downarrow) = (\psi\eta)(a).$$

Since $\psi\eta \in \mathcal{E}_\leqslant$, we conclude that $dzs \leqslant a$ in A. Consequently, $Dzs \subseteq a\!\downarrow$.

To complete the proof, we have to show that $D \circ z \subseteq D'$. Take $x \in D \circ z$. Since $D' = \mathsf{cl}(D')$, it suffices to prove that $x \in \mathsf{cl}(D')$, i.e.,

$$(\forall a \in A,\ s \in S)(D's \subseteq a\!\downarrow \Longrightarrow xs \leqslant a).$$

If $D's \subseteq a\!\downarrow$ then, by (3.1), we have $Dzs \subseteq a\!\downarrow$. Since $x \in D \circ z = \mathsf{cl}((Dz)\!\downarrow)$, we get $xs \leqslant a$ if we are able to prove that $(Dz)\!\downarrow\! s \subseteq a\!\downarrow$. If $d \in D,\ d' \in A$ and $d' \leqslant dz$ then $d's \leqslant dzs \leqslant a$, so $(Dz)\!\downarrow\! s \subseteq a\!\downarrow$, as needed. □

Example 15. Consider the additive pomonoid $S = (\mathbb{N}_0, +)$ of nonnegative integers acting on the set $A = \mathbb{N}$ by addition. For the S-poset A_S we have

$$\mathscr{P}(A) = \{n\!\downarrow \mid n \in \mathbb{N}\} \cup \{\emptyset, \mathbb{N}\} = \mathscr{Q}(A),$$

where $n\!\downarrow = \{1, \ldots, n\}$, and the action on $\mathscr{Q}(A)$ is defined by

$$\begin{aligned} n\!\downarrow \circ\, s &= \mathsf{cl}((n\!\downarrow + s)\!\downarrow) = \mathsf{cl}((n+s)\!\downarrow) = (n+s)\!\downarrow, \\ \mathbb{N} \circ s &= \mathsf{cl}((\mathbb{N} + s)\!\downarrow) = \mathsf{cl}(\mathbb{N}) = \mathbb{N}, \\ \emptyset \circ s &= \mathsf{cl}((\emptyset + s)\!\downarrow) = \emptyset, \end{aligned}$$

$s \in \mathbb{N}$. So $\mathscr{Q}(A)_S$ is isomorphic to $(\mathbb{N} \cup \{\theta, \infty\})_{\mathbb{N}_0}$ where $\infty + s = \infty$, $\theta + s = \theta$, and $\theta \leqslant a \leqslant \infty$ for all $s \in \mathbb{N}_0$ and $a \in \mathbb{N}$. In other words, we obtain the injective hull of A_S by adjoining external zero elements, one at the top, the other at the bottom.

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