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On descent theory for distributors

Valdis Laan and Pille Penjam

Abstract. We give necessary and sufficient conditions for equalizer preservation of the functor of tensor multiplication by a distributor and some sufficient conditions for a functor between small categories to be an effective descent functor.

1. Introduction

A functor $f : \mathcal{A} \to \mathcal{B}$ between small categories is called an effective descent functor if the so-called extension-of-scalars functor $f_! : \mathsf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set}) \longrightarrow \mathsf{Fun}(\mathcal{B}^{\mathrm{op}}, \mathsf{Set})$, induced by f, is comonadic. In this paper we give some sufficient conditions for f to be an effective descent functor. In Section 2 we give necessary and sufficient conditions for equalizer preservation for a more general situation than just for $f_!$. The results of Section 3 generalize the results of [6], where similar problems were considered for one-object categories (i.e. monoids) $\mathcal{A}$ and $\mathcal{B}$.

Throughout this paper, $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ will stand for small categories. By $\mathbf{1}$ we denote the discrete category with a single object $*$ and $\hat{\mathcal{A}} = \mathsf{Fun}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set})$. A *distributor* (or a *profunctor*; see e.g. [4]) from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\phi : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$. We write $\mathsf{Dist}(\mathcal{A}, \mathcal{B}) = \mathsf{Fun}(\mathcal{B}^{\mathrm{op}} \times \mathcal{A}, \mathsf{Set})$. By $(\hat{-}) : \mathsf{Fun}(\mathcal{B}^{\mathrm{op}} \times \mathcal{A}, \mathsf{Set}) \longrightarrow \mathsf{Fun}(\mathcal{A}, \hat{\mathcal{B}})$ and $(\overline{-}) : \mathsf{Fun}(\mathcal{A}, \hat{\mathcal{B}}) \longrightarrow \mathsf{Fun}(\mathcal{B}^{\mathrm{op}} \times \mathcal{A}, \mathsf{Set})$ we denote in the obvious way defined mutually inverse isomorphism functors.

Let $\phi : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$ be a distributor and $x \in \phi(B, A)$, $B \in \mathcal{B}, A \in \mathcal{A}$. If $a : A \to A'$ in $\mathcal{A}$ then we write $a \cdot x := \phi(1_B^{\mathrm{op}}, a)(x) \in \phi(B, A')$ and if $b : B' \to B$ in $\mathcal{B}$ then we write $x \cdot b := \phi(b^{\mathrm{op}}, 1_A)(x) \in \phi(B', A)$.

Consider now distributors $\phi : \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$ and $\psi : \mathcal{A}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$, and the Yoneda functor $\mathsf{Y}_{\mathcal{A}} : \mathcal{A} \to \hat{\mathcal{A}}$. Then $\hat{\phi} : \mathcal{A} \to \hat{\mathcal{B}}$ and there exists a left Kan extension $\mathsf{L}_{\mathsf{Y}_{\mathcal{A}}}(\hat{\phi}) : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ of $\hat{\phi}$ along $\mathsf{Y}_{\mathcal{A}}$ (this follows from the

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existence of small colimits in Set), which is denoted just $\mathsf{L}_{\mathcal{A}}(\hat{\phi})$. Thus the *composite* or *tensor product* of ψ and ϕ can be defined as the distributor

$$\psi \otimes \phi := \overline{\mathsf{L}_{\mathcal{A}}(\hat{\phi}) \circ \hat{\psi}} : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}.$$

$$\begin{tikzcd}
\hat{\mathcal{A}} \arrow[dr, "\mathsf{L}_{\mathcal{A}}(\hat{\phi})"] & \mathcal{A} \arrow[l, "\mathsf{Y}_{\mathcal{A}}"'] \arrow[d, "\hat{\phi}"] \\
\mathcal{C} \arrow[u, "\hat{\psi}"] \arrow[r, "\mathsf{L}_{\mathcal{A}}(\hat{\phi})\circ\hat{\psi}"'] & \hat{\mathcal{B}}
\end{tikzcd}$$

Note that for every $C \in \mathcal{C}, B \in \mathcal{B}$,

$$(\psi \otimes \phi)(B, C) = \frac{\coprod_{A \in \mathcal{A}} \psi(A, C) \times \phi(B, A)}{\sim}$$

is the quotient set by the smallest equivalence relation $\sim$ generated by all pairs $(x, y) \sim (x', y')$, $x \in \psi(A, C)$, $y \in \phi(B, A)$, $x' \in \psi(A', C)$, $y' \in \phi(B, A')$, such that

$$x = x' \cdot a \quad \text{and} \quad a \cdot y = y'$$

for some $a : A \to A'$ in $\mathcal{A}$. The last equalities can be illustrated by the "commutative" diagram

$$\begin{tikzcd}
& B \arrow[dl, dotted, "y"'] \arrow[dr, dotted, "y'"] & \\
A \arrow[rr, "a"] \arrow[dr, dotted, "x"'] & & A' \arrow[dl, dotted, "x'"] \\
& C &
\end{tikzcd}$$

where the dotted arrow labelled by y, for example, stands for the element y of $\phi(B, A)$. It is not difficult to see that $(x, y) \sim (x', y')$ if and only if there exists a "commutative" diagram

$$\begin{tikzcd}
& B \arrow[dl, dotted, "y"'] \arrow[dr, dotted, "y_1"] & & B \arrow[dl, dotted, "y_1"'] \arrow[dr, dotted, "y_2"] & & B \arrow[dl, dotted, "y_2"'] \arrow[dr, dotted, "y_3"] & & & & B \arrow[dl, dotted, "y_n"'] \arrow[dr, dotted, "y'"] & \\
A \arrow[dr, dotted, "x"'] & & A_1 \arrow[ll, "a_0"'] \arrow[rr, "a_1"] \arrow[dl, dotted, "x_1"] \arrow[dr, dotted, "x_1"'] & & A_2 \arrow[dl, dotted, "x_2"] \arrow[dr, dotted, "x_2"'] & & A_3 \arrow[ll, "a_2"'] \arrow[dl, dotted, "x_3"] & \cdots & A_n \arrow[rr, "a_n"] \arrow[dr, dotted, "x_n"'] & & A' \arrow[dl, dotted, "x'"] \\
& C & & C & & C & & & & C &
\end{tikzcd}$$

We denote the equivalence class of $(x, y) \in \psi(A, C) \times \phi(B, A)$ by $x \otimes_A y$, or just $x \otimes y$. So the basic rule for calculations is

$$x \cdot a \otimes_A y = x \otimes_{A'} a \cdot y \tag{1}$$

for every $a : A \to A'$ in $\mathcal{A}$, $x \in \psi(A', C)$, $y \in \phi(B, A)$.

For a fixed distributor $\phi \in \mathsf{Dist}(\mathcal{A},\mathcal{B})$ one may consider the functor $-\otimes\phi : \mathsf{Dist}(\mathcal{C},\mathcal{A}) \to \mathsf{Dist}(\mathcal{C},\mathcal{B})$ of *tensor multiplication by* ϕ, given by the assignment

$$\begin{tikzcd} \psi \arrow[r, mapsto] \arrow[d, "\mu"'] & \psi\otimes\phi \arrow[d, "\mu\otimes\phi"] \\ \psi' \arrow[r, mapsto] & \psi'\otimes\phi \end{tikzcd}$$

where the component $(\mu\otimes\phi)_{(B,C)} = \overline{\mathsf{L}_{\mathcal{A}}(\hat{\phi})\circ\hat{\mu}_{(B,C)}} : (\psi\otimes\phi)(B,C) \longrightarrow (\psi'\otimes\phi)(B,C)$ of the natural transformation $\mu\otimes\phi$ at $(B,C)\in\mathcal{B}^{\mathrm{op}}\times\mathcal{C}$ is the mapping given by

$$(\mu\otimes\phi)_{(B,C)}(k\otimes_A l) := \mu_{(A,C)}(k)\otimes_A l$$

where $A\in\mathcal{A}$ is such that $(k,l)\in\psi(A,C)\times\phi(B,A)$.

2. Equalizer flatness

The aim of this section is to obtain necessary and sufficient conditions for equalizer preservation of the functor $-\otimes\phi$, that will be applied in Section 3.

If $\mathcal{C} = \mathbf{1}$ then replacing $\mathsf{Dist}(\mathbf{1},\mathcal{A})$ and $\mathsf{Dist}(\mathbf{1},\mathcal{B})$ by isomorphic categories $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, we may assume that, for $\phi\in\mathsf{Dist}(\mathcal{A},\mathcal{B})$, $\psi\in\hat{\mathcal{A}}$, and $B\in\mathcal{B}$,

$$(\psi\otimes\phi)(B) = \frac{\coprod_{A\in\mathcal{A}}\psi(A)\times\phi(B,A)}{\sim} \tag{2}$$

is the quotient set by the smallest equivalence relation $\sim$ generated by all pairs $(x,y)\sim(x',y')$ such that

$$x = x'\cdot a = \psi(a^{\mathrm{op}})(x') \quad\text{ and }\quad \phi(1_B^{\mathrm{op}},a)(y) = a\cdot y = y'$$

for some $a : A\to A'$ in $\mathcal{A}$.

Note that two parallel morphisms $\mu,\nu : \psi\Rightarrow\chi$ in $\hat{\mathcal{A}}$ always have a *canonical equalizer* (α,ε), where

$$\begin{aligned} \alpha(A) &= \{x\in\psi(A)\mid \mu_A(x)=\nu_A(x)\},\\ \alpha(f)(x') &= \psi(f)(x') \end{aligned}$$

for every $A,A'\in\mathcal{A}$, $x'\in\psi(A)$ and $f : A\to A'$ in $\mathcal{A}^{\mathrm{op}}$, and $\varepsilon_A : \alpha(A)\to\psi(A)$ is the inclusion mapping for every $A\in\mathcal{A}$.

We shall need the following

Lemma 1. *Let* $\phi\in\mathsf{Dist}(\mathcal{A},\mathcal{B})$ *and* $\psi = \mathcal{A}^{\mathrm{op}}(A,-) = \mathcal{A}(-,A)\in\hat{\mathcal{A}}$, $A\in\mathcal{A}$. *Then* $a^{\mathrm{op}}\otimes_{A_0} y = (a')^{\mathrm{op}}\otimes_{A'} y'$ *in* $(\psi\otimes\phi)(B)$ *if and only if* $a\cdot y = a'\cdot y'$.

Proof. Necessity. The equality $a^{\mathrm{op}} \otimes_{A_0} y = (a')^{\mathrm{op}} \otimes_{A'} y'$ in $(\psi \otimes \phi)(B)$ means that there exists a "commutative" diagram

$$\begin{tikzcd}
& B \arrow[dl, dotted, "y"'] \arrow[dr, dotted, "y_1"] & & B \arrow[dl, dotted, "y_1"'] \arrow[dr, dotted, "y_2"] & & B \arrow[dl, dotted, "y_2"'] \arrow[dr, dotted, "y_3"] & & & & B \arrow[dl, dotted, "y_n"'] \arrow[dr, dotted, "y'"] & \\
A_0 \arrow[dr, "a"'] & & A_1 \arrow[ll, "a_0"'] \arrow[rr, "a_1"] \arrow[dl, "x_1"] \arrow[dr, "x_1"'] & & A_2 \arrow[dl, "x_2"] \arrow[dr, "x_2"'] & & A_3 \arrow[ll, "a_2"'] \arrow[dl, "x_3"] & \cdots & A_n \arrow[rr, "a_n"] \arrow[dr, "x_n"'] & & A' \arrow[dl, "a'"] . \\
& A & & A & & A & & & & A &
\end{tikzcd}$$

Hence

$$\begin{aligned}
a \cdot y &= a \cdot (a_0 \cdot y_1) = (a \circ a_0) \cdot y_1 = x_1 \cdot y_1 \\
&= (x_2 \circ a_1) \cdot y_1 = x_2 \cdot (a_1 \cdot y_1) = x_2 \cdot y_2 = \ldots \\
&= x_n \cdot y_n = (a' \circ a_n) \cdot y_n = a' \cdot (a_n \cdot y_n) = a' \cdot y'.
\end{aligned}$$

Sufficiency. If $a \cdot y = a' \cdot y'$ then $a^{\mathrm{op}} \otimes_{A_0} y = 1_A^{\mathrm{op}} \cdot a \otimes_{A_0} y = 1_A^{\mathrm{op}} \otimes_A a \cdot y = 1_A^{\mathrm{op}} \otimes_A a' \cdot y' = 1_A^{\mathrm{op}} \cdot a' \otimes_{A'} y' = (a')^{\mathrm{op}} \otimes_{A'} y'$. $\square$

The next theorem generalizes Proposition 1.1 of [3] from monoids to small categories.

Theorem 2. *For small categories $\mathcal{A}, \mathcal{B}$ and a distributor $\phi \in \mathsf{Dist}(\mathcal{A}, \mathcal{B})$, the following assertions are equivalent:*

1. *the functor $- \otimes \phi : \mathsf{Dist}(\mathcal{C}, \mathcal{A}) \to \mathsf{Dist}(\mathcal{C}, \mathcal{B})$ preserves equalizers for every small category $\mathcal{C}$;*
2. *the functor $- \otimes \phi : \mathsf{Dist}(\mathbf{1}, \mathcal{A}) \to \mathsf{Dist}(\mathbf{1}, \mathcal{B})$ preserves equalizers;*
3. *the functor $- \otimes \phi : \mathsf{Dist}(\mathbf{1}, \mathcal{A}) \to \mathsf{Dist}(\mathbf{1}, \mathcal{B})$ takes regular monomorphisms to monomorphisms, and for every $\chi \in \mathsf{Dist}(\mathbf{1}, \mathcal{A})$ and every $l \in \phi(B, A)$, $k, k' \in \chi(A)$, $A \in \mathcal{A}, B \in \mathcal{B}$, the equality $k \otimes_A l = k' \otimes_A l$ in $(\chi \otimes \phi)(B)$ implies that $l = a \cdot l'$ and $k \cdot a = k' \cdot a$ for some $a : A' \to A$ in $\mathcal{A}$ and $l' \in \phi(B, A')$.*

Proof. Obviously (1) $\Rightarrow$ (2). The implication (2) $\Rightarrow$ (1) holds because limits in functor categories are pointwise.

(3) $\Rightarrow$ (2). Assume that condition (3) is satisfied. Again, we identify $\mathsf{Dist}(\mathbf{1}, \mathcal{A})$ and $\mathsf{Dist}(\mathbf{1}, \mathcal{B})$ with $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$, respectively. Consider arbitrary $\psi, \chi \in \hat{\mathcal{A}}$ and $\mu, \nu : \psi \Rightarrow \chi$. It suffices to prove that the functor $- \otimes \phi$ preserves the canonical equalizer (α, ε) of (μ, ν). For this, we need to prove that the distributor $\alpha \otimes \phi$ is naturally isomorphic to the canonical equalizer (α', ε') of $(\mu \otimes \phi, \nu \otimes \phi)$ in $\hat{\mathcal{B}}$.

$$\begin{tikzcd}
\alpha \otimes \phi \arrow[rr, "\varepsilon \otimes \phi"] \arrow[dr, dashed, "\tau"'] & & \psi \otimes \phi \arrow[r, shift left, "\mu \otimes \phi"] \arrow[r, shift right, "\nu \otimes \phi"'] & \chi \otimes \phi \\
& \alpha' \arrow[ur, "\varepsilon'"'] & &
\end{tikzcd} \tag{3}$$

Note that, for every $B \in \mathcal{B}$, $(\alpha \otimes \phi)(B) = \frac{\coprod_{A \in \mathcal{A}} \alpha(A) \times \phi(B,A)}{\approx}$ and

$$\begin{aligned}
\alpha'(B) &= \{z \in (\psi \otimes \phi)(B) \mid (\mu \otimes \phi)_B(z) = (\nu \otimes \phi)_B(z)\} \\
&= \left\{ x \otimes_A y \in \frac{\coprod_{A \in \mathcal{A}} \psi(A) \times \phi(B,A)}{\sim} \mid \mu_A(x) \otimes_A y = \nu_A(x) \otimes_A y \right\},
\end{aligned}$$

where $\sim$ and $\approx$ are the relations defined as in (2). If $x \otimes_A y \in (\alpha \otimes \phi)(B)$ then $(x, y) \in \psi(A) \times \phi(B, A)$, and $\mu_A(x) \otimes_A y = \nu_A(x) \otimes_A y$, because

$$\mu_A(x) = \mu_A(\varepsilon_A(x)) = (\mu \circ \varepsilon)_A(x) = (\nu \circ \varepsilon)_A(x) = \nu_A(\varepsilon_A(x)) = \nu_A(x).$$

Hence we may define a mapping $\tau_B : (\alpha \otimes \phi)(B) \to \alpha'(B)$ by

$$\tau_B(x \otimes_A y) := x \otimes_A y.$$

It is straightforward to show that $\tau = (\tau_B)_{B \in \mathcal{B}^{\mathrm{op}}} : \alpha \otimes \phi \Rightarrow \alpha'$ is a natural transformation and the triangle in diagram (3) commutes. Since $\varepsilon \otimes \phi$ is a monomorphism by the assumption, we conclude that τ is a monomorphism.

To finish the proof, we show that each τ_B, $B \in \mathcal{B}$, is surjective (hence an isomorphism in Set, and thus τ is an isomorphism in $\hat{\mathcal{B}}$). Let $x \otimes_A y \in \alpha'(B)$, so $x \in \psi(A)$, $y \in \phi(B, A)$, and $\mu_A(x) \otimes_A y = \nu_A(x) \otimes_A y$ in $(\chi \otimes \phi)(B)$ for some $A \in \mathcal{A}$. By the assumption, there exist $a : A' \to A$ in $\mathcal{A}$ and $y' \in \phi(B, A')$ such that $y = a \cdot y'$ and $\chi(a^{\mathrm{op}})(\mu_A(x)) = \chi(a^{\mathrm{op}})(\nu_A(x))$. Now $\psi(a^{\mathrm{op}})(x) \in \psi(A')$ and

$$\mu_{A'}(\psi(a^{\mathrm{op}})(x)) = \chi(a^{\mathrm{op}})(\mu_A(x)) = \chi(a^{\mathrm{op}})(\nu_A(x)) = \nu_{A'}(\psi(a^{\mathrm{op}})(x))$$

mean that $\psi(a^{\mathrm{op}})(x) \in \alpha(A')$ and $\psi(a^{\mathrm{op}})(x) \otimes_{A'} y' \in (\alpha \otimes \phi)(B)$. Using property (1) we obtain

$$\begin{aligned}
\tau_B\left(\psi(a^{\mathrm{op}})(x) \otimes_{A'} y'\right) &= \psi(a^{\mathrm{op}})(x) \otimes_{A'} y' = x \cdot a \otimes_{A'} y' \\
&= x \otimes_A a \cdot y' = x \otimes_A y.
\end{aligned}$$

(2) $\Rightarrow$ (3). Assume that $- \otimes \phi$ preserves equalizers. Then it obviousy takes regular monomorphisms to monomorphisms. Suppose that $\chi \in \hat{\mathcal{A}}$, $l \in \phi(B, A)$, $k, k' \in \chi(A)$, $A \in \mathcal{A}$, $B \in \mathcal{B}$, are such that $k \otimes_A l = k' \otimes_A l$ in $(\chi \otimes \phi)(B)$. Consider the functor $\psi = \mathcal{A}^{\mathrm{op}}(A, -) : \mathcal{A}^{\mathrm{op}} \to \mathsf{Set}$, and, for every $A' \in \mathcal{A}$, define the mappings $\mu_{A'}, \nu_{A'} : \psi(A') \to \chi(A')$ by

$$\begin{aligned}
\mu_{A'}(a^{\mathrm{op}}) &:= \chi(a^{\mathrm{op}})(k) = k \cdot a, \\
\nu_{A'}(a^{\mathrm{op}}) &:= \chi(a^{\mathrm{op}})(k') = k' \cdot a,
\end{aligned}$$

$a : A' \to A$ in $\mathcal{A}$. Since

$$\begin{aligned}
\left(\chi(a_1^{\mathrm{op}}) \circ \mu_{A'}\right)(a^{\mathrm{op}}) &= \chi(a_1^{\mathrm{op}})(k \cdot a) = (k \cdot a) \cdot a_1 = k \cdot (a \circ a_1) \\
&= \mu_{A''}\left((a \circ a_1)^{\mathrm{op}}\right) = \mu_{A''}\left(a_1^{\mathrm{op}} \circ a^{\mathrm{op}}\right) \\
&= \left(\mu_{A''} \circ \psi(a_1^{\mathrm{op}})\right)(a^{\mathrm{op}})
\end{aligned}$$

for every $a : A' \to A$ and $a_1 : A'' \to A'$ in $\mathcal{A}$, $\mu : \psi \Rightarrow \chi$ (and analogously $\nu : \psi \Rightarrow \chi$) is a natural transformation.

$$\begin{tikzcd}
\mathcal{A}^{\mathrm{op}}(A,A') = \psi(A') \arrow[r, "\mu_{A'}"] \arrow[d, "a_1^{\mathrm{op}}\circ - = \psi(a_1^{\mathrm{op}})"'] & \chi(A') \arrow[d, "\chi(a_1^{\mathrm{op}}) = -\cdot a_1"] \\
\mathcal{A}^{\mathrm{op}}(A,A'') = \psi(A'') \arrow[r, "\mu_{A''}"'] & \chi(A'')
\end{tikzcd}$$

Let (α, ε) be the canonical equalizer of (μ, ν). By the assumption,

$$\begin{tikzcd}
\alpha \otimes \phi \arrow[r, "\varepsilon\otimes\phi"] & \psi \otimes \phi \arrow[r, shift left, "\mu\otimes\phi"] \arrow[r, shift right, "\nu\otimes\phi"'] & \chi \otimes \phi
\end{tikzcd}$$

is an equalizer diagram in $\hat{\mathcal{B}}$. If (α', ε') is the canonical equalizer of the pair $(\mu \otimes \phi, \nu \otimes \phi)$ and $B_1 \in \mathcal{B}$ then $(\alpha \otimes \phi)(B_1) = \frac{\bigsqcup_{A_1 \in \mathcal{A}} \alpha(A_1) \times \phi(B_1, A_1)}{\approx}$ and

$$\alpha'(B_1) = \left\{ x \otimes_{A_2} y \in \frac{\bigsqcup_{A_1 \in \mathcal{A}} \psi(A_1) \times \phi(B_1, A_1)}{\sim} \mid \mu_{A_2}(x) \otimes y = \nu_{A_2}(x) \otimes y \right\}.$$

If $x \otimes_{A_2} y \in (\alpha \otimes \phi)(B_1)$ then $y \in \phi(B_1, A_2)$ and $x \in \alpha(A_2)$. The last means that $x \in \psi(A_2)$ and $\mu_{A_2}(x) = \nu_{A_2}(x)$, so $\mu_{A_2}(x) \otimes_{A_2} y = \nu_{A_2}(x) \otimes_{A_2} y$ and $x \otimes_{A_2} y \in \alpha'(B_1)$. Therefore $(\alpha \otimes \phi)(B_1) \subseteq \alpha'(B_1)$ for every $B_1 \in \mathcal{B}$. Using the universal property of equalizers we conclude $\alpha \otimes \phi = \alpha'$. Now for the element

$$1_A^{\mathrm{op}} \otimes_A l \in \frac{\bigsqcup_{A_1 \in \mathcal{A}} \psi(A_1) \times \phi(B, A_1)}{\sim} = (\psi \otimes \phi)(B)$$

we calculate

$$\mu_A(1_A^{\mathrm{op}}) \otimes_A l = k \cdot 1_A \otimes_A l = k \otimes_A l = k' \otimes_A l = k' \cdot 1_A \otimes_A l = \nu_A(1_A^{\mathrm{op}}) \otimes_A l.$$

Hence $1_A^{\mathrm{op}} \otimes_A l \in \alpha'(B) = (\alpha \otimes \phi)(B)$, which means that $1_A^{\mathrm{op}} \otimes_A l = a^{\mathrm{op}} \otimes_{A_1} l'$ in $(\alpha \otimes \phi)(B)$ for some $l' \in \phi(B, A_1)$, $a : A_1 \to A$ in $\mathcal{A}$ such that $a^{\mathrm{op}} \in \alpha(A_1)$. The first equality implies by Lemma 1 that $l = a \cdot l'$ and the fact that $a^{\mathrm{op}} \in \alpha(A_1)$ implies that $k \cdot a = \mu_{A_1}(a^{\mathrm{op}}) = \nu_{A_1}(a^{\mathrm{op}}) = k' \cdot a$. $\square$

Remark 3. The second half of Condition (3) in Theorem 2 means that the existence of a "commutative" diagram

$$\begin{tikzcd}[column sep=small]
 & B \arrow[dl, dotted, "l"'] \arrow[dr, dotted, "l_1"] & & B \arrow[dl, dotted, "l_1"'] \arrow[dr, dotted, "l_2"] & & B \arrow[dl, dotted, "l_2"'] \arrow[dr, dotted, "l_3"] & & & & B \arrow[dl, dotted, "l_n"'] \arrow[dr, dotted, "l"] & \\
A \arrow[dr, dotted, "k"'] & & A_1 \arrow[ll, "a_0"'] \arrow[rr, "a_1"] \arrow[dl, dotted, "k_1"] \arrow[dr, dotted, "k_1"'] & & A_2 \arrow[dl, dotted, "k_2"] \arrow[dr, dotted, "k_2"'] & & A_3 \arrow[ll, "a_2"'] \arrow[dl, dotted, "k_3"] & \cdots & A_n \arrow[rr, "a_n"] \arrow[dr, dotted, "k_n"'] & & A' \arrow[dl, dotted, "k'"] \\
 & * & & * & & * & & & & * &
\end{tikzcd}$$

implies the existence of a "commutative" diagram

$$\begin{tikzcd}
 & B \arrow[dl, dotted, "l"'] \arrow[dr, dotted, "l'"] & & B \arrow[dl, dotted, "l'"'] \arrow[dr, dotted, "l"] & \\
A \arrow[dr, dotted, "k"'] & & A' \arrow[ll, "a"'] \arrow[rr, "a"] \arrow[dl, dotted, "k\cdot a"] \arrow[dr, dotted, "k\cdot a"'] & & A \arrow[dl, dotted, "k'"] \\
 & * & & * &
\end{tikzcd}$$

3. Descent functors and effective descent functors

First we recall some general results and definitions. Dualizing a part of Theorem 1, p. 138 of [7], we obtain

Theorem 4. *Let $\langle F, G; \eta, \varepsilon\rangle : \mathcal{Y} \to \mathcal{X}$ be an adjunction and $\mathbb{T} = \langle FG, \varepsilon, F\eta G\rangle$ the comonad it defines in $\mathcal{X}$. Then there is a (canonical) functor $K : \mathcal{Y} \to \mathcal{X}^{\mathbb{T}}$, where $\mathcal{X}^{\mathbb{T}}$ is the category of all $\mathbb{T}$-coalgebras.*

The dual of the following theorem of Beck can be found in [1], Theorem 3.9.

Theorem 5. *In the situation of Theorem 4, K is full and faithful if and only if η_Y is a regular monomorphism for every $Y \in \mathcal{Y}$.*

If K is full and faithful then F is called a *functor of descent type*. If K is an equivalence of categories then F is *comonadic* or of *effective descent type*.

We also shall use the following two results.

Lemma 6. *If $\langle F, G; \eta, \varepsilon\rangle : \mathcal{Y} \to \mathcal{X}$ is an adjunction and $F : \mathcal{Y} \to \mathcal{X}$ is of descent type then F reflects isomorphisms.*

Theorem 7. *Let $\mathcal{Y}, \mathcal{X}$ be categories with equalizers. A functor $F : \mathcal{Y} \to \mathcal{X}$ is comonadic if and only if*

(1) *F has a right adjoint;*
(2) *F reflects isomorphisms;*
(3) *F preserves equalizers of those pairs (h, g) for which $(F(h), F(g))$ is contractible.*

Now, let $f : \mathcal{A} \to \mathcal{B}$ be a functor. We denote $\phi_f = \overline{\mathsf{Y}_{\mathcal{B}} \circ f} = \mathcal{B}(-, f(-)) \in \mathsf{Dist}(\mathcal{A}, \mathcal{B})$. Then the restriction-of-scalars functor $f_* = - \circ f^{\mathrm{op}} : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ has a left adjoint $f_! = \mathsf{L}_{\mathcal{A}}(\mathsf{Y}_{\mathcal{B}} \circ f) : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ (extension-of-scalars), which is isomorphic to the functor $- \otimes \phi_f : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ (see [2], Section 6.3).

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathsf{Y}_{\mathcal{A}}"] \arrow[d, "f"'] & \hat{\mathcal{A}} \arrow[d, shift right=2, "\mathsf{L}_{\mathcal{A}}(\mathsf{Y}_{\mathcal{B}} \circ f)"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{B} \arrow[r, "\mathsf{Y}_{\mathcal{B}}"'] & \hat{\mathcal{B}} \arrow[u, shift right=2, "- \circ f^{\mathrm{op}}"']
\end{tikzcd}$$

Definition 8. A functor $f : \mathcal{A} \to \mathcal{B}$ is called a *descent functor* (an *effective descent functor*), if $f_!$ is of descent type (respectively, comonadic).

Note that the unit $\eta : 1_{\hat{\mathcal{A}}} \Rightarrow f_* \circ f_!$ of the adjunction $f_! \dashv f_*$ is the natural transformation defined for every $\psi \in \hat{\mathcal{A}}$ by

$$\begin{aligned} (\eta_\psi)_A : \psi(A) &\to ((\psi \otimes \phi_f) \circ f^{\mathrm{op}})(A) = \frac{\coprod_{A' \in \mathcal{A}} \psi(A') \times \mathcal{B}(f(A), f(A'))}{\sim}, \\ x &\mapsto x \otimes_A 1_{f(A)}, \end{aligned}$$

$A \in \mathcal{A}, x \in \psi(A)$.

Proposition 9. *A functor $f : \mathcal{A} \to \mathcal{B}$ is a descent functor if and only if $(\eta_\psi)_A$ is an injective mapping for every $\psi \in \hat{\mathcal{A}}$ and $A \in \mathcal{A}$.*

Proof. For $\psi \in \hat{\mathcal{A}}$, η_ψ is a regular monomorphism if and only if all $(\eta_\psi)_A$, $A \in \mathcal{A}$, are regular monomorphisms in Set (i.e. injective mappings). Hence the result follows from Theorem 5. □

Corollary 10. *Descent functors are faithful.*

Proof. Consider a descent functor $f : \mathcal{A} \to \mathcal{B}$ and morphisms $a, a' : A \to A_0$ in $\mathcal{A}$ such that $f(a) = f(a')$. With $\psi = \mathcal{A}^{\mathrm{op}}(A_0, -) \in \hat{\mathcal{A}}$ we calculate

$$\begin{aligned} a^{\mathrm{op}} \otimes_A 1_{f(A)} &= 1_{A_0}^{\mathrm{op}} \cdot a \otimes_A 1_{f(A)} = 1_{A_0}^{\mathrm{op}} \otimes_{A_0} a \cdot 1_{f(A)} \\ &= 1_{A_0}^{\mathrm{op}} \otimes_{A_0} f(a) \circ 1_{f(A)} = 1_{A_0}^{\mathrm{op}} \otimes_{A_0} f(a) = 1_{A_0}^{\mathrm{op}} \otimes_{A_0} f(a') \\ &= 1_{A_0}^{\mathrm{op}} \otimes_{A_0} f(a') \circ 1_{f(A)} = 1_{A_0}^{\mathrm{op}} \cdot a' \otimes_A 1_{f(A)} = (a')^{\mathrm{op}} \otimes_A 1_{f(A)} \end{aligned}$$

in $(\psi \otimes \phi_f)(f(A))$. Since $(\eta_\psi)_A$ is injective, $a = a'$. □

Now we give some sufficient conditions for f to be an effective descent functor. By Theorem 7, f is an effective descent functor, if the functor $f_! : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ reflects isomorphisms and preserves all equalizers. Specializing Theorem 2 to ϕ_f we obtain

Proposition 11. *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor. The functor $- \otimes \phi_f : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ preserves equalizers if and only if*

1. *it takes regular monomorphisms to monomorphisms, and*
2. *for every $\chi \in \hat{\mathcal{A}}$ and every $l \in \mathcal{B}(B, f(A))$, $k, k' \in \chi(A)$, $A \in \mathcal{A}, B \in \mathcal{B}$, the equality $k \otimes_A l = k' \otimes_A l$ in $(\chi \otimes \phi_f)(B)$ implies that $l = f(a) \circ l'$ and $\chi(a^{\mathrm{op}})(k) = \chi(a^{\mathrm{op}})(k')$ for some $a : A' \to A$ in $\mathcal{A}$ and $l' \in \mathcal{B}(B, f(A'))$.*

Proposition 12. *Let $f : \mathcal{A} \to \mathcal{B}$ be a functor. If f reflects split epimorphisms and the functor $- \otimes \phi_f : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ preserves equalizers then f is an effective descent functor.*

Proof. Suppose that $x \otimes_A 1_{f(A)} = x' \otimes_A 1_{f(A)}$ in $(\psi \otimes \phi_f)(f(A))$, $A \in \mathcal{A}$, $\psi \in \hat{\mathcal{A}}$, $x, x' \in \psi(A)$. By Proposition 11, $1_{f(A)} = f(a) \circ l'$ and $\psi(a^{\mathrm{op}})(x) = \psi(a^{\mathrm{op}})(x')$ for some $a : A' \to A$ in $\mathcal{A}$ and $l' : f(A) \to f(A')$ in $\mathcal{B}$. Hence $a \circ a' = 1_A$ for some $a' : A \to A'$ in $\mathcal{A}$. Consequently,

$$x = \psi(1_A^{\mathrm{op}})(x) = \psi((a')^{\mathrm{op}} \circ a^{\mathrm{op}})(x) = \psi((a')^{\mathrm{op}} \circ a^{\mathrm{op}})(x') = \psi(1_A^{\mathrm{op}})(x') = x',$$

which means that $(\eta_\psi)_A$ is injective for every $\psi \in \hat{\mathcal{A}}$ and $A \in \mathcal{A}$. By Proposition 9, f is a descent functor and by Lemma 6, $f_!$ reflects isomorphisms. The result now follows from Theorem 7. $\square$

Recall that a functor $f : \mathcal{A} \to \mathcal{B}$ is *flat* if the functor $- \otimes \phi_f : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ preserves finite limits. Similarly we say that a functor $f : \mathcal{A} \to \mathcal{B}$ is *pullback flat (equalizer flat)* if the functor $- \otimes \phi_f : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ preserves pullbacks (equalizers). Since pullback flatness implies equalizer flatness, by Proposition 12 we have the following implications for f:

flat and reflects split epis $\Longrightarrow$ pullback flat and reflects split epis
$\Longrightarrow$ equalizer flat and reflects split epis $\Longrightarrow$ effective descent functor
$\Longrightarrow$ descent functor $\Longrightarrow$ faithful.

Using the fact that the left Kan extension $\mathsf{L}_{\mathcal{A}}(F) : \hat{\mathcal{A}} \to \mathsf{Set}$ of a functor $F : \mathcal{A} \to \mathsf{Set}$ preserves pullbacks if and only if the category of elements of F is co-pseudofiltered (see [7], p. 212), as in Theorem 6.4 of [2] one can see that a functor $f : \mathcal{A} \to \mathcal{B}$ is pullback flat if and only if the category $B \downarrow f$ is co-pseudofiltered for every object $B \in \mathcal{B}$.

Corollary 13. *If f reflects split epimorphisms and the category $B \downarrow f$ is co-pseudofiltered for every object $B \in \mathcal{B}$ then f is an effective descent functor.*

Corollary 14. *Every faithful functor between groupoids is an effective descent functor.*

Proof. Suppose that $f : \mathcal{A} \to \mathcal{B}$ is a faithful functor between groupoids. If $f(a_1) \circ b_1 = b_2 = f(a_2) \circ b_1$ in the right hand side of the diagram

$$\begin{tikzcd}
& B \arrow[dl, dashed, "b_1"'] \arrow[d, "b_1"] \arrow[dr, "b_2"] & \\
f(A_1) \arrow[r, dashed, "f(1_{A_1})"'] & f(A_1) \arrow[r, shift left, "f(a_1)"] \arrow[r, shift right, "f(a_2)"'] & f(A_2)
\end{tikzcd}$$

then $f(a_1) = f(a_2)$ and $a_1 = a_2$. Hence we can complete the diagram with dotted arrows. The rest of the proof is illustrated by the diagram

$$\begin{tikzcd}
 & f(A_1) \arrow[dr, "f(a_1)"] & \\
f(A_2) \arrow[ur, dashed, "f(a_1^{-1} \circ a_2)"] \arrow[dr, dashed, "f(1_{A_2})"'] & B \arrow[l, dashed, "b_2"'] \arrow[u, "b_1"'] \arrow[r, "b"] \arrow[d, "b_2"] & f(A) \,. \\
 & f(A_2) \arrow[ur, "f(a_2)"'] &
\end{tikzcd}$$

□

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University of Tartu, Institute of Mathematics, J. Liivi 2, 50409 Tartu, Estonia

E-mail address: `valdis.laan@ut.ee`

University of Tartu, Institute of Estonian and General Linquistics, Ülikooli 18, 50090 Tartu, Estonia

E-mail address: `pille.penjam@ut.ee`