# On flat acts*

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Let $S$ be a monoid. A set $A$ is called a *right $S$-act* if, for every element $a$ from $A$ and every element $s$ from $S$, their product $as$, belonging to the set $A$, is defined in such a way that $a1 = a$ and $(as_1)s_2 = a(s_1s_2)$ for any element $a$ of $A$ and any elements $s_1, s_2$ from $S$. Analogously, left $S$-acts are defined (see [4]). There are some properties of monoids which are determined by the properties of left (right) acts over the corresponding monoids. Some of such properties are considered in the work [4], where a description of those monoids is given, over which all left acts are injective, but also a description of monoids, over which all left acts are projective.

In this note, tensor product of acts is defined, and it is used to introduce the notion of a flat left act. It is known that all left (right) modules over a ring are flat if and only if this ring is regular (see [3], Theorem 1). It turns out that in the case of monoids and acts, the corresponding fact is not true. Although from the flatness of all left $S$-acts the regularity of the monoid $S$ follows (Theorem 1), Lemma 3 implies that the converse is not true. As Theorem 2 shows, this fact still holds in the class of left cancellative monoids. Theorem 3 asserts that if all right acts over a monoid are injective, then all left acts over that monoid are flat. Finally, it is shown that there exist monoids, over which all left acts are flat, but the same does not hold for right acts.

Let $S$ be a monoid, $A$ a right $S$-act and $M$ a left $S$-act. On the direct product $A \times M$ we give a relation $\Theta$ by putting $(a_1, m_1)\Theta(a_2, m_2)$, where $a_1, a_2 \in A$, $m_1, m_2 \in M$, if and only if there exists a finite chain of pairs from $A \times M$ such that the first pair of the chain coincides with $(a_1, m_1)$, the last pair coincides with $(a_2, m_2)$ and from each previous pair to the next pair one can go with the help of moving an element of $S$, i.e., going from a pair $(as, m)$, where $s \in S$, to the pair $(a, sm)$, or vice versa. The

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relation $\Theta$, defined in this way, will be an equivalence relation. We call the quotient set $A \times M/\Theta$ the *tensor product of acts $A$ and $M$* and denote it by $A \otimes_S M$ or $A \otimes M$. The equivalence class, to which the pair $(a, m)$ belongs, will be denoted by $a \otimes m$.

We have

**Lemma 1.** *Let $S$ be a monoid and $M$ an arbitrary left $S$-act. Then the tensor product $S \otimes M$ of a monoid $S$, considered as a right $S$-act, and the left $S$-act $M$ is equivalent to the set $M$, i.e. $S \otimes_S M \cong M$.*

*Proof* is analogous to the proof of the corresponding result for modules over a ring (see, for example, [1], Theorem 12.14). We only note here that this equivalence is realized by the mapping $\varphi : S \otimes_S M \to M$, which is defined by the equality $\varphi(s \otimes m) = sm$, where $s \in S$, $m \in M$.

It is known that in the category of left $S$-acts direct sums coincide with disjoint unions. Therefore we have

**Lemma 2.** *Tensor product of $S$-acts commutes with direct sums.*

Let $A_1$ and $A_2$ be right $S$-acts, $M_1$ and $M_2$ left $S$-acts, $\varphi : A_1 \to A_2$ and $\psi : M_1 \to M_2$ $S$-homomorphisms. The homomorphisms $\varphi$ and $\psi$ induce the following mapping $\varphi \otimes \psi$ from the tensor product $A_1 \otimes M_1$ to the tensor product $A_2 \otimes M_2$: $(\varphi \otimes \psi)(a_1 \otimes m_1) = \varphi(a_1) \otimes \psi(m_1)$ for all $a_1 \in A_1$ and $m_1 \in M_1$.

We will call a left $S$-act $M$ *flat* if, from the fact that $A$ is a subact of a right $S$-act $B$ and the elements $a_1 \otimes m_1$ and $a_2 \otimes m_2$, where $a_1, a_2 \in A$, $m_1, m_2 \in M$, are different in the tensor product $A \otimes M$, it always follows that the elements $a_1 \otimes m_1$ and $a_2 \otimes m_2$ are different also in the tensor product $B \otimes M$.

**Theorem 1.** *If all left $S$-acts are flat, then $S$ is regular.*

*Proof.* Let all left $S$-acts be flat. To prove the theorem, it suffices to show that every element $s$ of $S$ belongs to the set $sSs$. If $sS = S$ or $Ss = S$, then this is obviously so. Because of that, we will assume that the inclusions $sS \subset S$ and $Ss \subset S$ are strict. Further, we denote by $S'$ the left $S$-act obtained from $S$ by gluing together the elements of $Ss$. We will denote the elements of $S'$ as follows: if $s \in S$, then we will denote the corresponding element in $S'$ by $s'$, if $s \notin Ss$, and by $0'$, if $s \in Ss$. Consider now the elements $s \otimes 1'$ and $s \otimes 0'$ in the tensor product $sS \otimes S'$.

These elements must be equal in $sS \otimes S'$, because $S'$ is a flat left $S$-act by assumption and in $S \otimes S'$ we have $s \otimes 1' = 1 \otimes s1' = 1 \otimes 0'$. Consequently, there exists a finite chain of pairs, leading from the pair $(s, 1')$ to the pair $(s, 0')$, where all the intermediate pairs belong to $sS \times S'$, i.e., they have the form $(su, v')$ for some $u, v \in S$. We will make a convention to consider the first pair having the second component $0'$ as the last pair in the chain.

We will prove that for all pairs $(su, v')$ in our chain, except the last pair, the equality $suv = s$ holds. Clearly, this equality holds for the first pair. Let it hold for the pair $(su, v')$, which strictly precedes the penultimate pair. We will prove that a similar equality holds for the next pair. For the transition from the pair $(su, v')$ to the next pair there are two possibilities: a) $su = skt$ for some $k, t \in S$ and the next pair will be $(sk, tv') = (sk, (tv)')$, where $sktv = suv = s$, so that in this case the needed equality indeed holds; b) $kw' = v'$ for some $k, w \in S$ and the next pair will be $(suk, w')$. Since $v' \neq 0'$, also $w' \neq 0'$. Hence $kw = v$ and $sukw = suv = s$, so that also in this case the needed equality holds. Now let $(su, v')$ be the penultimate pair. We know that $suv = s$. The last transition will move an element of $S$ from the left to the right. That means, $su = sxy$ for some $x, y \in S$ and the last pair will be $(sx, yv')$. According to our convention, $(yv)' = yv' = 0'$, i.e. $yv = zs$ for some $z \in S$. Hence

$$s = suv = sxyv = sxzs \in sSs.$$

The proof is complete.

**Theorem 2.** *Let $S$ be a left cancellative monoid. All left $S$-acts are flat if and only if $S$ is a group.*

*Proof.* In [4], it is proved that a monoid $S$ is a group if and only if each left (right) $S$-act is a disjoint union of cyclic left (right) $S$-acts. From this result, from Lemma 2 and from the fact that a cyclic left (right) act over a group does not have proper subacts, it follows that if the monoid $S$ is a group, then all left $S$-acts are flat.

Conversely, if all left $S$-acts are flat, then Theorem 1 implies that $S$ is regular, from where, due to left cancellability, it follows that $S$ is a group. The proof is complete.

Over each monoid $S$ there exists a one-element left $S$-act, which we will call the zero $S$-act and denote by 0 (as well as its only element).

**Lemma 3.** *If all left $S$-acts are flat, then every two right ideals of the monoid $S$ have a non-empty intersection.*

*Proof.* Let $a$ and $b$ be elements of $S$ such that $aS \cap bS = \emptyset$. Then $aS \cup bS = I \neq S$. Let us consider the elements $a \otimes 0$ and $b \otimes 0$ in the tensor product $I \otimes 0$. These elements are different in the tensor product $I \otimes 0$, because with the help of moving arbitrary elements of $S$, from the pair $(a, 0)$ we always obtain a pair whose first component belongs to $aS$. Flatness of the zero act now implies that $a \otimes 0 \neq b \otimes 0$ in the tensor product $S \otimes 0$, which consists of a single element $1 \otimes 0$. This is a contradiction. The proof is complete.

Now we can give examples of (completely) regular monoids, over which not all left acts are flat.

*Example.* Let $S = 1 \cup S'$, where $S'$ is a non-singleton left zero semigroup. Then $S$ is a (completely) regular semigroup. On the other hand, every single element of $S'$ forms a principal right ideal of the monoid $S$. Lemma 3 implies that the zero left $S$-act cannot be flat.

**Theorem 3.** *If all right acts over a monoid $S$ are injective, then all left $S$-acts are flat.*

*Proof.* Let the assumptions of the theorem be satisfied, let $B$ be an arbitrary right $S$-act, $A$ its arbitrary subact and $M$ any left $S$-act. Further, let, for some $a_1, a_2 \in A$ and $m_1, m_2 \in M$, the equality $a_1 \otimes m_1 = a_2 \otimes m_2$ hold in the tensor product $B \otimes M$. Since $A$ is a subact of the act $B$, the injectivity of $A$ implies that $A$ is a retract of $B$, i.e. there exists an $S$-homomorphism $\varphi : B \to A$ such that $\varphi(a) = a$ for all $a \in A$. Then $\varphi \otimes 1_M$ is a mapping from the tensor product $B \otimes M$ to $A \otimes M$ and, in $A \otimes M$, we have

$$\begin{aligned} a_1 \otimes m_1 = \varphi(a_1) \otimes m_1 = (\varphi \otimes 1_M)(a_1 \otimes m_1) = \\ = (\varphi \otimes 1_M)(a_2 \otimes m_2) = \varphi(a_2) \otimes m_2 = a_2 \otimes m_2. \end{aligned}$$

Since $a_1, a_2, m_1$ and $m_2$ were chosen arbitrarily, this yields that $M$ is flat. The proof is complete.

**Corollary.** *Let $S$ be a commutative monoid with zero, all ideals of which are principal. Then all acts over $S$ are flat if and only if $S$ is regular.*

*Proof.* If all $S$-acts are flat, then regularity of $S$ follows from Theorem 1. If $S$ is regular, then Theorem 2 in [4] implies that all $S$-acts are injective. From Theorem 3 it follows that all left $S$-acts are flat.

In what follows we will need the following

**Proposition.** *If a monoid $S$ has form $S = 1 \cup S'$, where $1 \notin S'$ and $S'$ is a right simple semigroup, then every right $S$-act is a disjoint union of indecomposable subacts $A_\gamma$, $\gamma \in I$, where $I$ is some index set, such that $A_\gamma$ is described by the following condition: for any elements $a_1$ and $a_2$ of $A_\gamma$, the set $a_1S'$ is a subact of the act $A_\gamma$ and $a_1S' = a_2S'$.*

*Proof.* Let $S = 1 \cup S'$, where $1 \notin S'$, $S'$ is a right simple semigroup and let $A$ be an arbitrary right $S$-act. For every element $a \in A$, the set $aS'$ is a subact of the act $A$. On the act $A$ we define a relation $\equiv$ by putting $a_1 \equiv a_2 \iff a_1S' = a_2S'$. We denote the classes of this equivalence relation by $A_\gamma$, $\gamma \in I$, where $I$ is a set of indices. The relation $(as)S' = aS'$, which holds for all $a \in A$ and $s \in S$, implies that all $A_\gamma$, $\gamma \in I$, are subacts of the act $A$. Their indecomposability follows from the definition of the relation of $\equiv$.

**Theorem 4.** *If a monoid $S$ has form $S = 1 \cup S'$, where $1 \notin S'$ and $S'$ is a right simple semigroup containing idempotents, then all left $S$-acts are flat.*

*Proof.* Let $B$ be any right $S$-act, $A$ its subact and $M$ a left $S$-act. In view of Lemma 2 and the previous proposition, we can assume that $B$ is an indecomposable act. We denote by $e_\gamma$, $\gamma \in I$, all elements of $B$ having the following property: if $e_\gamma = e_\gamma s$, $s \in S$, then $s = 1$. The previous proposition implies that $e_\gamma S' = bS'$ for every $b \in B$ and $\gamma \in I$. Therefore the subact $A$ contains $e_\gamma S'$ for every $\gamma \in I$ and $B \setminus A$ contains only the elements $e_\gamma$, $\gamma \in I$ (not necessarily all of them)[1]. Let now $a_1, a_2 \in A$ and $m_1, m_2 \in M$ be elements such that $a_1 \otimes m_1 = a_2 \otimes m_2$ in the tensor product $B \otimes M$ and suppose that at the same time $a_1 \otimes m_1 \neq a_2 \otimes m_2$ in $A \otimes M$. Let us fix a chain of pairs leading from the pair $(a_1, m_1)$ to the pair $(a_2, m_2)$ by moving the elements from $S$. Note that, since $a_1 \otimes m_1 \neq a_2 \otimes m_2$ in $A \otimes M$, our chain must contain a transition, which will take us outside $A \times M$. The transition which follows it, must return us to $A \times M$, because

[1] If $b \notin \{e_\gamma \mid \gamma \in I\}$, then there exists $s \in S'$ such that $b = bs$ and hence $b = bs \in bS' = e_\gamma S' \subseteq A$. (Added by translator.)

in the case of an element from $(B \setminus A) \times M$ a 'move right' transition is impossible, but every 'move left' transition will return the first component to $e_i S' \subset A$. Suppose now that the first transition will take us outside $A \times M$ and already the second will take us to the pair $(a_2, m_2)$. In that case $a_1 = e_{\gamma_0} x$ and $a_2 = e_{\gamma_0} y$, where $e_{\gamma_0} \in B \setminus A$, $x, y \in S'$ and $(e_{\gamma_0}, xm_1) = (e_{\gamma_0}, ym_2)$, i.e. $xm_1 = ym_2$. By the assumption of the theorem, $S'$ has an idempotent $e$ which (see [2], page 313) is a left identity of the semigroup $S'$. Consequently, $ex = x$, $ey = y$ and $(a_1, m_1) = ((e_{\gamma_0} e)x, m_1)$. By moving $x$ to the right from the last pair we obtain $(e_{\gamma_0} e, xm_1) = (e_{\gamma_0} e, ym_2)$. Now moving $y$ to the left we obtain $((e_{\gamma_0} e)y, m_2) = (e_{\gamma_0} y, m_2) = (a_2, m_2)$. Since $e_{\gamma_0} e \in A$, this means that we can move from the pair $(a_1, m_1)$ to the pair $(a_2, m_2)$ staying in $A \times M$. Therefore, in this case we have arrived at a contradiction with the fact that $a_1 \otimes m_1 \neq a_2 \otimes m_2$ in $A \otimes M$. We will arrive at the same contradiction also in the general case. Thus, in fact $a_1 \otimes m_1 = a_2 \otimes m_2$ must also hold in $A \otimes M$. Since $a_1, a_2, m_1$ and $m_2$ were chosen arbitrarily, this follows that every left $S$-act $M$ is flat. The proof is complete.

**Corollary.** *There exist monoids over which all left acts are flat, but the same in untrue for right acts.*

*Proof.* Since a right zero semigroup is right simple, our proposition implies that all left $S$-acts are flat if $S$ is a monoid of the form $S = 1 \cup S'$, where $1 \notin S'$ and $S'$ is a non-singleton right zero semigroup. At the same time the right hand version of Lemma 3 shows that not all right $S$-acts are flat (see the example after Lemma 3).

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## Literature

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## A note from translator

This article was published in 1970 in the journal *Tartu Riikliku Ülikooli Toimetised* (Proceedings of the Tartu State University), which was the predecessor of the journal *Acta et Commentationes Universitatis Tartuensis de Mathematica*, where I am currently the Editor-in-Chief. This article is important because it is here where flat acts over monoids were defined (meaning that a right act $A_S$ over a monoid $S$ is *flat* if the tensor functor $A\otimes_S - : {}_S\mathsf{Act} \to \mathsf{Set}$ preserves monomorphisms) and the first results about them were proved. This created a new research direction in the theory of acts over monoids – the study of flatness properties (including weak flatness, principal weak flatness and others). Since 1970, a large number of semigroup theorists has contributed to this area, including myself as an academic son of Mati Kilp.

At that time, scientific articles in Estonia had to be published in Russian language, which made them largely inaccessible for the most of the world's scientific community. Since I find this article a very important milestone, I decided to translate it into English to make it readable for all interested mathematicians. During translation, some typos have also been corrected.

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