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PULLBACKS AND FLATNESS PROPERTIES OF ACTS

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CONTENTS

INTRODUCTION

For all algebraic structures, studying their representations is a natural part of the theory of these structures. The same holds for semigroups and monoids. Among representations of monoids, representations by transformations of sets are probably most important. Such a representation may be considered as an algebraic structure (with unary operations corresponding to the elements of this monoid) which will be later called an act over a monoid, so we can speak about the theory of acts over monoids instead of the theory of representations of monoids by transformations of sets. Any module over a ring (with identity) is an example of an act over a monoid.

There are many different properties of acts that have been investigated. Special emphasis has been on properties having categorical origin, such as projectivity, injectivity, freeness, cofreeness and so on. These properties can be divided, roughly speaking, into two big groups: properties gathered around projectivity and properties gathered around injectivity. In this work we consider only projectivity and related properties.

A large number of results about acts over monoids concerns so-called homological classification of monoids by properties of acts over them. That means the consideration of questions like "Which conditions must a monoid satisfy in order for all acts over it with one property to have another property?". Before acts over monoids, similar questions were asked for modules over rings. It turned out that the situation for acts is much different from the situation for modules. Namely, many so-called flatness properties (which are generalizations of projectivity) are the same for modules but essentially different for acts.

In this work we try to classify monoids by flatness properties of acts. The earliest works in this area belong to Kilp and by now a lot of articles have been published on this topic. Among flatness properties, pullback flatness (or strong flatness, which is the most common term) is the strongest. It was introduced in [32] under the name of weak flatness. A right act over a monoid is called pullback flat if tensoring by it preserves all pullback diagrams in the category of left acts over this monoid. From the properties under consideration pullback flatness implies condition (P), flatness, weak flatness, principal weak flatness and torsion freeness. From [2] it follows that, besides pullback flatness, also condition (P) can be described in terms of tensoring of pullbacks. So we can pose the question: can the other weaker flatness properties be characterized in the same way and are these properties the only ones which can be obtained in this way? We want to see what happens if we require preserving only all pullback diagrams of a certain type or if we do not require preserving but something less.

We start with a scheme where, by dropping requirements, we obtain a formal structure of properties which an act can or cannot have. Closer examination in section 2 shows which of these properties are actually different. We see that condition (P), flatness, weak flatness, principal weak flatness and torsion freeness find their place in this scheme and, moreover, we see new properties emerging. We use the terms 'weakly pullback flat', 'weakly homoflat' and 'principally weakly homoflat' to denote these properties.

In section 3 we try to solve some classification problems. Although flatness properties do not form a chain with respect to order of decreasing strength, for homological classification purposes it is natural to choose such a chain. We have chosen here a chain starting with pullback flatness, ending with torsion freeness and including all 'new' properties (and leaving flatness and weak flatness aside). Some of the results in section 3 concern only 'old' properties (for example we find a description of monoids over which all torsion free right acts are principally weakly flat) but most of them involve 'new' properties, too. The results obtained are tabulated at the end.

Since the conditions on acts or monoids might be quite complicated, we quite often use the standard symbols of mathematical logic hoping that this will not cause confusion, but rather help to avoid it.

1 PRELIMINARIES

Throughout this paper let S denote a monoid with an identity element 1. We start with basic definitions of the theory of acts over monoids.

Definition 1 A nonempty set A is called a *right S-act* (or a *right act over S*) and denoted A_S (or simply A if the context permits to drop S) if S acts on A unitarily from the right, that is, there exists a mapping $A \times S \to A$, $(a, s) \mapsto as$, satisfying the conditions

1. $(as)t = a(st)$,

2. $a1 = a$

for all $a \in A$ and all $s, t \in S$. Left S-acts ${}_SA$ are defined dually.

Definition 2 A nonempty subset B of a right (left) S-act A is called a *subact* of A if $bs \in B$ ($sb \in B$) for all $b \in B$ and $s \in S$.

Definition 3 An equivalence relation ρ on a right (left) S-act A is called a *congruence* on A if $a_1 \rho a_2$ implies $(a_1 s)\rho(a_2 s)$ $((sa_1)\rho(sa_2))$ for all $a_1, a_2 \in A$ and $s \in S$.

Definition 4 If A and B are right (left) S-acts then a mapping $f : A \to B$ is called a *homomorphism* of right (left) S-acts if

$$f(as) = f(a)s$$

$(f(sa) = sf(a))$ for all $a \in A$ and $s \in S$.

A nonempty subset K of a monoid S is a *right (left) ideal* of S, if $ks \in K$ ($sk \in K$) for all $s \in S$ and $k \in K$. A right (left) ideal K of a monoid S is called *proper*, if it is not equal to S. It is called *trivial*, if $K = S$ or $|K| = 1$. By a *trivial monoid* we mean the one-element monoid.

Every right (left) ideal K of S is in a natural way a right (left) S-act which is a subact of S_S (${}_SS$).

An equivalence relation ρ on a monoid S is a *right (left) congruence* on S if $x_1 \rho x_2$ implies $(x_1 s)\rho(x_2 s)$ $\quad ((sx_1)\rho(sx_2))$ for all $x_1, x_2, s \in S$. A right and left congruence on S is called a *congruence* on S.

So congruences of the right S-act S_S are exactly right congruences of the monoid S.

The notion of tensor product plays important role in the study of acts over monoids. The definition of tensor product of acts was first given in [12].

Definition 5 ([12]) If A_S and ${}_SB$ are a right and a left S-act, respectively, the *tensor product* $A_S \otimes {}_SB$ of A_S and ${}_SB$ (over the monoid S) is the quotient set $(A \times B)/\tau$, where τ is the smallest equivalence relation on $A \times B$ that identifies all pairs (as, b) and (a, sb), $a \in A_S, b \in {}_SB, s \in S$.

The τ-class of a pair $(a, b) \in A \times B$ is denoted by $a \otimes b$, so for every $s \in S$ we have the equality

$$as \otimes b = a \otimes sb.$$

For calculation purposes we shall use the following lemma, which actually can be formulated in several different ways (see [4], [20]).

Lemma 1.1 ([10]) *Let S be a monoid, $a, a' \in A_S, b, b' \in {}_SB$. Then $a \otimes b = a' \otimes b'$ in $A_S \otimes {}_SB$ if and only if there exist a natural number n and elements $a_1, \ldots, a_{n-1} \in A_S, b_1, \ldots, b_{n-1} \in {}_SB$, $s_1, \ldots, s_n$, $t_1, \ldots, t_{n-1} \in S$ such that*

$$\begin{array}{rclrcl} a & = & a_1s_1 & s_1b & = & t_1b_1 \\ a_1t_1 & = & a_2s_2 & s_2b_1 & = & t_2b_2 \\ & \ldots & & & \ldots & \\ a_{n-1}t_{n-1} & = & a's_n & s_nb_{n-1} & = & b'. \end{array}$$

A sequence of equalities as in Lemma 1.1 is called a *scheme* (or *tossing*) of length n over A_S and ${}_SB$ joining (a, b) and (a', b').

The following two lemmas are direct consequences of Lemma 1.1.

Lemma 1.2 *Let S be a monoid, $a, a' \in A_S, b, b' \in {}_SB$. Then $a \otimes b = a' \otimes b'$ in $A_S \otimes_S (Sb \cup Sb')$ if and only if there exist a natural number n and elements $a_1, \ldots, a_{n-1} \in A_S, b_1, \ldots, b_{n-1} \in \{b, b'\}$, $s_1, \ldots, s_n$, $t_1, \ldots, t_{n-1} \in S$ such that*

$$\begin{array}{rclrcl} a & = & a_1s_1 & s_1b & = & t_1b_1 \\ a_1t_1 & = & a_2s_2 & s_2b_1 & = & t_2b_2 \\ & \ldots & & & \ldots & \\ a_{n-1}t_{n-1} & = & a's_n & s_nb_{n-1} & = & b'. \end{array}$$

Lemma 1.3 ([12]) *Let S be a monoid. Then $a \otimes s = a' \otimes t$ in $A_S \otimes {}_SS$ if and only if $as = a't$ in A_S.*

Fix a right S-act A_S. Let us show that this gives rise to a functor $A_S \otimes_S -$ from the category of all left S-acts to the category of sets. For objects ${}_SM$ of the category of left S-acts let $A_S \otimes_S -$ be defined by

$${}_SM \mapsto A_S \otimes {}_SM$$

and for morphisms $f{:}{}_SM \to {}_SN$ in the category of left S-acts (that is for homomorphisms of left S-acts) by

$$f \mapsto \mathrm{id}_A \otimes f$$

where

$$\mathrm{id}_A \otimes f : A_S \otimes {}_SM \to A_S \otimes {}_SN$$

is defined by

$$(\mathrm{id}_A \otimes f)(a \otimes m) = a \otimes f(m)$$

for all $a \in A_S$ and $m \in {}_SM$ (it follows from Lemma 1.1 that $\mathrm{id}_A \otimes f$ is well-defined). Then

$$(A_S \otimes_S -)(\mathrm{id}_{_SM}) = \mathrm{id}_{A_S} \otimes \mathrm{id}_{_SM} = \mathrm{id}_{A_S \otimes {}_SM} = \mathrm{id}_{(A_S \otimes_S -)({}_SM)}$$

because

$$(\mathrm{id}_{A_S} \otimes \mathrm{id}_{_SM})(a \otimes m) = a \otimes \mathrm{id}_{_SM}(m) = a \otimes m = \mathrm{id}_{A_S \otimes {}_SM}(a \otimes m)$$

for all $a \in A_S$, $m \in {}_SM$. Take two homomorphisms $f{:}{}_SM \to {}_SN$, $g{:}{}_SN \to {}_SQ$ of left S-acts. Then

$$\begin{aligned}(\mathrm{id}_{A_S} \otimes gf)(a \otimes m) &= a \otimes (gf)(m) = a \otimes g(f(m)) \\ &= (\mathrm{id}_{A_S} \otimes g)(a \otimes f(m)) \\ &= (\mathrm{id}_{A_S} \otimes g)((\mathrm{id}_{A_S} \otimes f)(a \otimes m)) \\ &= ((\mathrm{id}_{A_S} \otimes g)(\mathrm{id}_{A_S} \otimes f))(a \otimes m)\end{aligned}$$

for all $a \in A_S$, $m \in {}_SM$. This means that

$$\mathrm{id}_{A_S} \otimes gf = (\mathrm{id}_{A_S} \otimes g)(\mathrm{id}_{A_S} \otimes f),$$

or

$$(A_S \otimes_S -)(gf) = (A_S \otimes_S -)(g)(A_S \otimes_S -)(f).$$

Hence $A_S \otimes_S -$ is indeed a covariant functor. This functor is called the *functor of tensoring* by A_S.

We now give the definitions of properties of S-acts related to flatness that have been under closer examination in many articles on homological classification of monoids.

Definition 6 ([12]) A right S-act A_S is called *flat* if the functor $A_S \otimes_S -$ preserves all monomorphisms.

Thus A_S is flat if for every monomorphism $\iota{:}{}_SM \to {}_SN$ the mapping $\mathrm{id}_A \otimes \iota : A_S \otimes {}_SM \to A_S \otimes {}_SN$ is injective, that is $a \otimes \iota(m) = a' \otimes \iota(m')$ implies $a \otimes m = a' \otimes m'$ for all $a, a' \in A_S$, $m, m' \in {}_SM$. The most often used (already since [12]) reformulation of the definition is the following one.

Lemma 1.4 *A right S-act A_S is flat if and only if for every left S-act ${}_SN$, its subact ${}_SM$ and all elements $a, a' \in A_S$, $m, m' \in {}_SM$, if $a \otimes m$ and $a' \otimes m'$ are equal in the tensor product $A_S \otimes {}_SN$ then they are equal already in the tensor product $A_S \otimes {}_SM$.*

Definition 7 ([15]) A right S-act A_S is called *(principally) weakly flat* if the functor $A_S \otimes_S -$ preserves all monomorphisms from (principal) left ideals of S into S.

From definitions of weak flatness and principal weak flatness and Lemma 1.3 we get the following criteria for checking weak flatness and principal weak flatness.

Lemma 1.5 *A right S-act A_S is weakly flat if and only if for all $a, a' \in A_S$, $s, t \in S$, if $as = a't$ then $a \otimes s = a' \otimes t$ in the tensor product $A_S \otimes {}_S(Ss \cup St)$.*

Lemma 1.6 *A right S-act A_S is principally weakly flat if and only if for all $a, a' \in A_S$, $s, t \in S$, if $as = a's$ then $a \otimes s = a' \otimes s$ in the tensor product $A_S \otimes {}_S(Ss)$.*

It follows from the definitions that flatness implies weak flatness and weak flatness implies principal weak flatness. As proved in [15], the following definition gives a generalization of principal weak flatness.

Definition 8 ([24]) A right S-act A_S is called *torsion free* if $ac = a'c$ implies $a = a'$ for all $a, a' \in A_S$ and right cancellable $c \in S$.

The following definitions give some properties which turn out to be stronger than flatness.

Definition 9 A right S-act A_S is called *free* if there exists a subset $U \subseteq A_S$ such that every element $a \in A_S$ can be uniquely presented in the form $a = us$, $u \in U$, $s \in S$, i.e., if $a = u_1s_1 = u_2s_2$, $u_1, u_2 \in U$, $s_1, s_2 \in S$ then $u_1 = u_2$ and $s_1 = s_2$. The subset U is called a *basis* of A_S.

Projectivity in the category of right acts is defined as in every category.

Definition 10 A right S-act A_S is called *projective* if for every epimorphism $\pi : P_S \to Q_S$ and every homomorphism $f : A_S \to Q_S$ there exists a homomorphism $g : A_S \to P_S$ such that $\pi g = f$.

From Definition 9 the following description of freeness follows (note that $\coprod$ denotes the disjoint union).

Theorem 1.7 *A right S-act A_S is free if and only if $A_S \cong \coprod_{i\in I} A_i$, where $A_i \cong S_S$ for every $i \in I$.*

For projectivity the following description is known.

Theorem 1.8 ([22]) *A right S-act A_S is projective if and only if $A_S \cong \coprod_{i\in I} A_i$, where for every $i \in I$ there exists an idempotent $e_i \in S$ such that $A_i \cong (e_iS)_S$.*

From Theorems 1.7 and 1.8 it immediately follows that every free S-act is projective.

Pullback diagrams in the category of left S-acts (or sets) are defined as in every category.

Definition 11 A diagram

$$\begin{tikzcd} {}_SP \arrow[r, "p_1"] \arrow[d, "p_2"'] & {}_SM \arrow[d, "f"] \\ {}_SN \arrow[r, "g"'] & {}_SQ \end{tikzcd}$$

where ${}_SP, {}_SM, {}_SN$ and ${}_SQ$ are left S-acts and f, g, p_1 and p_2 are homomorphisms of left S-acts, is called a *pullback diagram*, or a *pullback square,* if $fp_1 = gp_2$ and for every left S-act ${}_S\overline{P}$, all homomorphisms $\overline{p}_1 : {}_S\overline{P} \to {}_SM$ and $\overline{p}_2 : {}_S\overline{P} \to {}_SN$ such that $f\overline{p}_1 = g\overline{p}_2$ there exists a unique homomorphism $h: {}_S\overline{P} \to {}_SP$ such that $p_1h = \overline{p}_1$ and $p_2h = \overline{p}_2$.

Every nonempty set can be considered as a left act over a trivial monoid. Homomorphisms of such left acts are just mappings of sets and so with the previous definition, pullback diagrams in the category of sets are defined, too.

We omit the definition of equalizer diagram since we shall not use it in what follows. Interested readers can see [30], for example.

Definition 12 ([32]) A right S-act A_S is called *strongly flat* if the functor $A_S \otimes_S -$ preserves pullbacks and equalizers.

The meaning of the word 'preserves' will be explained in subsection 2.1.

Definition 13 ([30]) A right S-act A_S is called *pullback flat* if the functor $A_S \otimes_S -$ preserves pullbacks.

Definition 14 ([30]) A right S-act A_S is called *equalizer flat* if the functor $A_S \otimes_S -$ preserves equalizers.

Definition 15 ([32]) A right S-act A_S satisfies *condition* (P) if

$$(\forall a, a' \in A_S)(\forall s, s' \in S)(as = a's' \Rightarrow$$
$$(\exists a'' \in A_S)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs')).$$

Definition 16 ([32]) A right S-act A_S satisfies *condition* (E) if

$$(\forall a \in A_S)(\forall s, s' \in S)(as = as' \Rightarrow$$
$$(\exists a' \in A_S)(\exists u \in S)(a = a'u \wedge us = us')).$$

Originally, acts for which the functor of tensoring preserves equalizers and pullbacks, were called weakly flat in [32] and it was proved there that the functor of tensoring by a right S-act preserves equalizers and pullbacks if and only if this act satisfies conditions (P) and (E). Afterwards such acts have been called strongly flat starting from [13]. In [13] it was also proved that strong flatness implies flatness. It turned out that flatness is essentially weaker than strong flatness, namely it was proved in [19] that all flat right S-acts are strongly flat if and only if S has only one element. In [30] the interval between flatness and strong flatness was investigated in detail. In this paper pullback flat and equalizer flat acts (as well as acts satisfying condition (P) or (E)) were first considered on their own and it was shown that condition (P) implies flatness. This made condition (P) a suitable intermediate property between flatness and strong flatness for homological classification purposes. However, the question whether pullback flatness implies strong flatness remained open. In [2] it was proved that an act is pullback flat if and only if it is strongly flat. This means that pullback flatness implies equalizer flatness. For our purposes we shall use the term 'pullback flat' as a synonym of 'strongly flat'. So for the pullback flatness we have the following description.

Theorem 1.9 ([32], [2]) *A right S-act is pullback flat if and only if it satisfies conditions* (P) *and* (E)*.*

It was shown in [32] that projectivity implies pullback flatness. So we have the following implications:

freeness $\Rightarrow$ projectivity $\Rightarrow$ pullback flatness $\Rightarrow$ condition (P) $\Rightarrow$
$\Rightarrow$ flatness $\Rightarrow$ weak flatness $\Rightarrow$
$\Rightarrow$ principal weak flatness $\Rightarrow$ torsion freeness.

There exist examples in the literature showing that all these implications are strict.

Every cyclic right S-act (i.e. an act which is generated by a single element) is isomorphic to a factor act S/ρ where ρ is a right congruence on S. A ρ-class of an element $s \in S$ will be denoted by $[s]_\rho$ or simply $[s]$.

For cyclic acts we shall need the following proposition.

Proposition 1.10 ([1], [7], [23]) *Let ρ be a right congruence on a monoid S. A cyclic right S-act S/ρ*

- *satisfies condition* (P) *if and only if*

$$(\forall s, t \in S)(s\rho t \Rightarrow (\exists u, v \in S)(us = vt \wedge u\rho 1 \wedge v\rho 1));$$

- *is pullback flat (satisfies condition* (E)*) if and only if*

$$(\forall s, t \in S)(s\rho t \Rightarrow (\exists u \in S)(us = ut \wedge u\rho 1));$$

- *is torsion free if and only if*

$$(\forall s, t, c \in S)(sc\rho tc \wedge \ c \textit{ is right cancellable } \Rightarrow s\rho t).$$

If $K \subseteq S$ is a right ideal of S then the binary relation ρ_K on S, defined by

$$s\rho_K t \iff ((s = t) \vee (s, t \in K)),$$

$s, t \in S$, is a right congruence on S. The factor act S/ρ_K will be denoted S/K and it is called a *right Rees factor act* of S by K.

For a monoid S we can consider any one-element set $\Theta = \{\theta\}$ as a right S-act Θ_S by defining

$$\theta s = \theta$$

for all $s \in S$. Since all such one-element right S-acts are isomorphic, we can speak about *the one-element right S-act* Θ_S. From the previous paragraph we have

$$S/S \cong \Theta_S.$$

For right Rees factor acts and Θ_S the descriptions of flatness properties take simpler forms. Before formulating them we need some more definitions.

Definition 17 ([17]) A monoid S is called *left (right) collapsible* if for every $s, s' \in S$ there exists $z \in S$ such that $zs = zs'$ ($sz = s'z$).

Definition 18 A monoid S is called *right (left) reversible* if for every $s, s' \in S$ there exist $u, v \in S$ such that $us = vs'$ ($su = s'v$).

Definition 19 A right ideal K of a monoid S is called *left stabilizing* if for every $k \in K$ there exists $l \in K$ such that $lk = k$.

Left stabilizing right ideals came up in [15], the name was first given in [3]

Proposition 1.11 *Let K be a right ideal of a monoid S. A right Rees factor act S/K*

- *is (weakly) flat if and only if S is right reversible and K is left stabilizing* [15]*;*
- *is principally weakly flat if and only if K is left stabilizing* [15]*;*
- *is torsion free if and only if $sc \in K$ implies $s \in K$ for $s, c \in S$, c right cancellable* [20]*;*
- *satisfies condition* (P) *if and only if $|K| = 1$ or $K = S$ and S is right reversible* [15]*;*
- *is pullback flat (satisfies condition* (E)*) if and only if $|K| = 1$ or $K = S$ and S is left collapsible* [20]*;*
- *is projective if and only if $|K| = 1$ or $K = S$ and S has a left zero* [20]*;*
- *is free if and only if $|K| = 1$* [20].

Corollary 1.12 *The one-element right S-act Θ_S*

- *is (weakly) flat if and only if S is right reversible;*
- *is principally weakly flat;*
- *is torsion free;*
- *satisfies condition* (P) *if and only if S is right reversible;*
- *is pullback flat (satisfies condition* (E)*) if and only if S is left collapsible;*
- *is projective if and only if S has a left zero;*
- *is free if and only if $S = \{1\}$.*

The following lemmas will be used when working with factor acts.

Lemma 1.13 ([29]) *Let A_S be a right S-act, $H \subseteq A_S \times A_S$ and $\rho(H)$ the smallest congruence on A_S, which contains H. Then $(a, a') \in \rho(H)$, $a, a' \in A_S$, if and only if either $a = a'$ or there exist a natural number n and elements $y_1, \ldots, y_n \in S$, $b_1, \ldots, b_n, d_1, \ldots, d_n \in A_S$ such that*

$$\begin{array}{rclrcl} a &=& b_1 y_1 & d_2 y_2 &=& b_3 y_3 \\ & & d_1 y_1 = b_2 y_2 & & & \ldots\ d_n y_n = a', \end{array}$$

where either $(b_i, d_i) \in H$ or $(d_i, b_i) \in H$ for every $i \in \{1, \ldots, n\}$.

We shall write simply $\rho(b, d)$ instead of $\rho(\{(b, d)\})$. As special cases of the previous lemma we get the following lemmas.

Lemma 1.14 *Let A_S be a right S-act, $b, d \in A_S$ and $\rho(b, d)$ the smallest congruence on A_S, which contains the pair (b, d). Then $(a, a') \in \rho(b, d)$, $a, a' \in A_S$, if and only if either $a = a'$ or there exist a natural number n and elements $y_1, \ldots, y_n \in S$, $b_1, \ldots, b_n, d_1, \ldots, d_n \in A_S$ such that*

$$\begin{array}{rclrcl} a &=& b_1 y_1 & d_2 y_2 &=& b_3 y_3 \\ & & d_1 y_1 = b_2 y_2 & & & \ldots\ d_n y_n = a', \end{array}$$

where $\{b_i, d_i\} = \{b, d\}$ for every $i \in \{1, \ldots, n\}$.

Lemma 1.15 *Let S be a monoid, $s, t \in S$ and $\rho(s, t)$ the smallest right congruence on S, which contains the pair (s, t). Then $(x, y) \in \rho(s, t)$, $x, y \in S$, if and only if either $x = y$ or there exist a natural number n and elements $y_1, \ldots, y_n$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in S$ such that*

$$\begin{array}{rclrcl} x &=& s_1 y_1 & t_2 y_2 &=& s_3 y_3 \\ & & t_1 y_1 = s_2 y_2 & & & \ldots\ t_n y_n = y, \end{array}$$

where $\{s_i, t_i\} = \{s, t\}$ for every $i \in \{1, \ldots, n\}$.

Let S be a monoid and let ${}_S S_1$ and ${}_S S_2$ be two left S-acts, which are isomorphic to the left S-act ${}_S S$. Then there exist left S-act isomorphisms $\alpha_1 : {}_S S \to {}_S S_1$ and $\alpha_2 : {}_S S \to {}_S S_2$. For the images of an element s of S we shall write $\alpha_1(s) = s_1$ and $\alpha_2(s) = s_2$. Thus, for instance, 1_1 and 1_2 are the isomorphic copies of the identity element 1 of S in S_1 and S_2, respectively, and using that α_1 and α_2 are homomorphisms of left S-acts we have

$$(st)_i = st_i,$$

$s, t \in S$, $i \in \{1, 2\}$. In what follows we shall also make use of the left S-act ${}_S(S_1 \bigsqcup S_2)$ which is just the disjoint union of S_1 and S_2 on which S acts in the natural way from the left, that is

$$\begin{array}{rcl} xs_1 &=& (xs)_1 \\ xs_2 &=& (xs)_2 \end{array}$$

for all $x, s \in S$.

If S is a monoid and $t \in S$ then $\lambda_t : S \to S$ ($\rho_t : S \to S$) will denote the left (right) translation by t, i.e.

$$\lambda_t(s) = ts$$

($\rho_t(s) = st$) for every $s \in S$. Then $\ker \lambda_t$ ($\ker \rho_t$) is a right (left) congruence on S.

Let T be a semigroup. Taking $S = T \sqcup \{1\}$ the disjoint union of T and $\{1\}$ and defining

$$11 = 1 \text{ and } t1 = 1t = t$$

for every $t \in T$ we obtain a monoid S with the identity element 1. We say that the monoid S is obtained from T by (external) *adjoining of identity*. This monoid is denoted by T^1.

Let T be a monoid. Taking $S = T \sqcup \{0\}$ and defining

$$t0 = 0t = 00 = 0$$

for every $t \in T$ we get a monoid S with a zero element 0. We say that the monoid S is obtained from T by (external) *adjoining of zero* and denote it by T^0.

2 TENSORING OF PULLBACKS AND FLATNESS PROPERTIES

2.1 A new look at the tensoring of pullbacks

Consider a diagram

$$\begin{tikzcd} {}_SP \arrow[r, "p_1"] \arrow[d, "p_2"'] & {}_SM \arrow[d, "f"] \\ {}_SN \arrow[r, "g"'] & {}_SQ \end{tikzcd} \qquad \text{(P1)}$$

in the category of left S-acts.

Lemma 2.1 ([30], [31]) *If* (P1) *is a pullback diagram then* ${}_SP$ *is isomorphic to the left* S*-act* $\{(m,n) \in {}_SM \times {}_SN \mid f(m) = g(n)\}$ *where* $s(m,n) = (sm, sn)$ *for all* $s \in S$, $m \in {}_SM$ *and* $n \in {}_SN$.

So if (P1) is a pullback of homomorphisms f and g then ${}_SP$ is determined up to isomorphism and we may assume that it is

$${}_SP = \{(m,n) \in {}_SM \times {}_SN \mid f(m) = g(n)\},$$

and p_1, p_2 are the restrictions of the projections, that is, $p_1((m,n)) = m$ and $p_2((m,n)) = n$ for every $(m,n) \in {}_SP$. With this convention let us denote such a pullback diagram (P1) in the category of left S-acts by $P(M,N,f,g,Q)$.

In the same way one can construct the pullback of two mappings in the category of sets, because nonempty sets can be considered as left acts over a trivial monoid.

Tensoring the pullback diagram $P(M,N,f,g,Q)$ by any right S-act A_S one gets the commutative diagram

$$\begin{tikzcd} A_S \otimes {}_SP \arrow[r, "\mathrm{id}_A \otimes p_1"] \arrow[d, "\mathrm{id}_A \otimes p_2"'] & A_S \otimes {}_SM \arrow[d, "\mathrm{id}_A \otimes f"] \\ A_S \otimes {}_SN \arrow[r, "\mathrm{id}_A \otimes g"'] & A_S \otimes {}_SQ \end{tikzcd}$$

in the category of sets.

For the pullback of mappings $\mathrm{id}_A \otimes f$ and $\mathrm{id}_A \otimes g$ in the category of sets we may take by our convention

$$P' = \{(a \otimes m, a' \otimes n) \in (A_S \otimes {}_SM) \times (A_S \otimes {}_SN) \mid a \otimes f(m) = a' \otimes g(n)\}$$

with ${p_1}'$ and ${p_2}'$ the restrictions of the projections. (Note that the existence of the pullback diagram (P1) implies the existence of the pullback diagram of $\mathrm{id}_A \otimes f$ and $\mathrm{id}_A \otimes g$.)

Now it follows from the definition of pullbacks that there exists a unique mapping $\varphi : A_S \otimes {}_SP \longrightarrow P'$ such that the diagram

$$\begin{tikzcd}
A_S \otimes {}_SP \arrow[dr, "\varphi"] \arrow[drr, "\mathrm{id}_A \otimes p_1"] \arrow[ddr, "\mathrm{id}_A \otimes p_2"'] & & \\
& P' \arrow[r, "p_1'"'] \arrow[d, "p_2'"] & A_S \otimes {}_SM \arrow[d, "\mathrm{id}_A \otimes f"] \\
& A_S \otimes {}_SN \arrow[r, "\mathrm{id}_A \otimes g"'] & A_S \otimes {}_SQ
\end{tikzcd} \qquad \text{(P2)}$$

is commutative. We shall call this mapping *the mapping φ corresponding to the pullback diagram $P(M, N, f, g, Q)$.*

It was stated in [2] that the mapping φ in Diagram (P2) is given by

$$\varphi(a \otimes (m, n)) = (a \otimes m, a \otimes n)$$

for all $a \in A_S$ and $(m, n) \in {}_SP$.

Note that surjectivity of φ means that

$$(\forall a, a' \in A_S)(\forall m \in {}_SM)(\forall n \in {}_SN)[a \otimes f(m) = a' \otimes g(n) \Rightarrow$$
$$(\exists a'' \in A_S)(\exists m' \in {}_SM)(\exists n' \in {}_SN)$$
$$(f(m') = g(n') \wedge a \otimes m = a'' \otimes m' \wedge a' \otimes n = a'' \otimes n')]$$

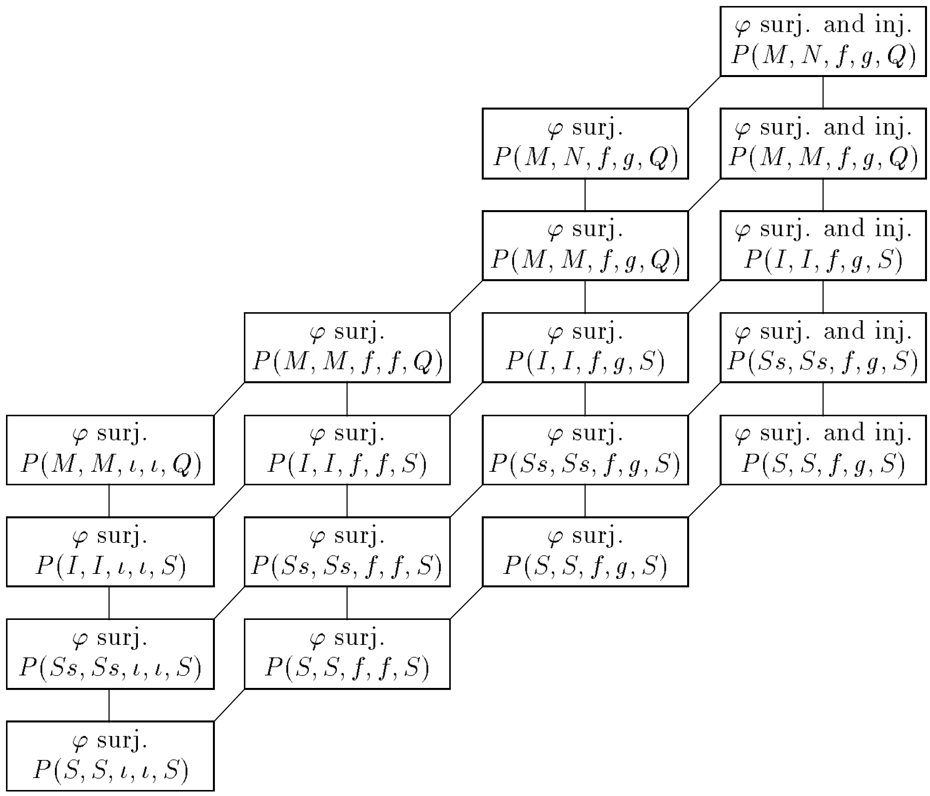
and injectivity of φ means that

$$(\forall a, a' \in A_S)(\forall m, m' \in {}_SM)(\forall n, n' \in {}_SN)$$
$$[f(m) = g(n) \wedge f(m') = g(n') \wedge a \otimes m = a' \otimes m' \wedge a \otimes n = a' \otimes n' \Rightarrow$$
$$a \otimes (m, n) = a' \otimes (m', n') \text{ in } A_S \otimes {}_SP].$$

In what follows, if we want to use surjectivity or injectivity of φ corresponding to some pullback diagram, we always use these statements, without specially emphasizing it.

By the original definition, pullback flatness of A_S means that the corresponding φ is surjective and injective for every pullback diagram $P(M, N, f, g, Q)$ in the category of left S-acts. It was shown in [2] that this holds if and only if A_S satisfies conditions (E) and (P). Moreover, in [2] it was proved that an act A_S satisfies condition (P) if and only if the corresponding φ is surjective for every pullback diagram $P(M, N, f, g, Q)$. So we notice that if we drop the requirement of injectivity of φ in the definition of pullback flatness, we obtain condition (P). On the other hand we know that condition (P) implies flatness. So the following questions arise: Can we get flatness by weakening the requirements that φ has to fulfill in order for A_S to satisfy condition (P)? What happens if we demand surjectivity of φ not for all pullback diagrams but only for some specific kind of them? And so on.

It turns out that it is indeed possible to give for flatness (and its generalizations) a description that uses surjectivity of φ for some special pullbacks. Of course, there are several ways how one can weaken requirements on φ to get possible generalizations of pullback flatness. What we have chosen as our aim here is to generalize pullback flatness so that the well-known flatness properties result and to find out whether there exist any new properties that can be obtained by making steps towards generalization similar to those which we need in order to get condition (P) from pullback flatness, flatness from condition (P), weak flatness from flatness and so on. Having this in mind, it seems reasonable to study what happens if we consider pullbacks of two equal homomorphisms, if this homomorphism is actually a monomorphism or if we restrict ourselves only to homomorphisms (monomorphisms) from (principal) left ideals of S to S. What results is the following rather formal collection of properties that can be organized into the following scheme.



Scheme 1

Here I (Ss) stands for a (principal) left ideal of S, and ι for a monomorphism of left S-acts. Every rectangle stands for a class of right S-acts that is defined by the property written into it in short but, we hope, understandably. For instance, a rectangle with the text "φ surj. and inj. $P(Ss, Ss, f, g, S)$" denotes the class of all right S-acts A_S such that the corresponding φ is surjective and injective for every pullback diagram $P(Ss, Ss, f, g, S)$, $s \in S$. A line between two rectangles indicates that the class of right S-acts corresponding to the rectangle at the upper end of the line is contained in the class corresponding to the rectangle at the lower end.

A priori we do not know whether these classes coincide. The rest of this section is devoted to determing which of these classes are actually different. We try to give descriptions of the corresponding properties, which do not use the notion of pullback. For new properties (those which do not appear in section 1) we also give conditions under which cyclic acts, Rees factor acts, or one-element act have this property.

2.2 On torsion free acts

First we see that the lowest cell in the leftmost column of Scheme 1 is actually the class of torsion free right S-acts.

Proposition 2.2 *A right S-act A_S is torsion free if and only if the corresponding φ is surjective for every pullback diagram $P(S,S,\iota,\iota,S)$, where $\iota : {}_SS \to {}_SS$ is a monomorphism of left S-acts.*

Proof. **Necessity.** Let A_S be torsion free. Suppose that $a \otimes \iota(s) = a' \otimes \iota(t)$ for some $s,t \in S$ and a monomorphism $\iota : {}_SS \to {}_SS$ of left S-acts. Since ι is a homomorphism of left S-acts, $a \otimes s\iota(1) = a' \otimes t\iota(1)$ in $A_S \otimes {}_SS$. By Lemma 1.3 this means that $as\iota(1) = a't\iota(1)$ in A_S. From the injectivity of ι it follows that the element $\iota(1) \in S$ is right cancellable. Hence torsion freeness of A_S implies that $as = a't$. Denote $a'' = as = a't$. Then we have $a \otimes s = a'' \otimes 1$ and $a' \otimes t = a'' \otimes 1$ in $A_S \otimes {}_SS$ and $\iota(1) = \iota(1)$. Thus φ is surjective for the pullback diagram $P(S,S,\iota,\iota,S)$.

Sufficiency. Suppose that $ac = a'c$ for some $a, a' \in A_S$ and right cancellable $c \in S$. Take a mapping $\iota = \rho_c : {}_SS \to {}_SS$. Since c is right cancellable, ι is a monomorphism of left S-acts. Now $a\iota(1) = a'\iota(1)$, which means that $a \otimes \iota(1) = a' \otimes \iota(1)$ in $A_S \otimes {}_SS$ by Lemma 1.3. Surjectivity of φ for the diagram $P(S,S,\iota,\iota,S)$ implies that there exist $a'' \in A_S$, $s', t' \in S$ such that $\iota(s') = \iota(t')$, $a \otimes 1 = a'' \otimes s'$ and $a' \otimes 1 = a'' \otimes t'$ in $A_S \otimes {}_SS$. Injectivity of ι implies $s' = t'$ and Lemma 1.3 implies $a = a''s'$ and $a' = a''t'$. Thus $a = a'$ and A_S is torsion free. ∎

2.3 On principally weakly flat acts

It turns out that the third cell from the top in the leftmost column of Scheme 1 is the class of principally weakly flat right S-acts.

Proposition 2.3 *A right S-act A_S is principally weakly flat if and only if the corresponding φ is surjective for every pullback diagram $P(Ss, Ss, \iota, \iota, S)$, where $s \in S$ and $\iota : {}_S(Ss) \to {}_SS$ is a monomorphism of left S-acts.*

Proof. **Necessity.** Let A_S be principally weakly flat. Suppose that $a \otimes \iota(us) = a' \otimes \iota(vs)$ for some $a, a' \in A_S, s, u, v \in S$ and a monomorphism $\iota : {}_S(Ss) \to {}_SS$ of left S-acts. Principal weak flatness of S implies that $a \otimes us = a' \otimes vs$ in $A_S \otimes {}_S(Ss)$. Thus φ is surjective for the pullback diagram $P(Ss, Ss, \iota, \iota, S)$, because $a \otimes us = a' \otimes vs$ and $a' \otimes vs = a' \otimes vs$ in $A_S \otimes {}_S(Ss)$ and $\iota(vs) = \iota(vs)$.

Sufficiency. Suppose that $a \otimes \iota(us) = a' \otimes \iota(vs)$ in $A_S \otimes {}_SS$ for some $a, a' \in A_S$, $u, v, s \in S$ and a monomorphism $\iota : {}_S(Ss) \to {}_SS$. Surjectivity of φ for the diagram $P(Ss, Ss, \iota, \iota, S)$ implies that there exist $a'' \in A_S$,

$u', v' \in Ss$ such that $\iota(u') = \iota(v')$, $a \otimes us = a'' \otimes u'$ in $A_S \otimes {}_S(Ss)$ and $a' \otimes vs = a'' \otimes v'$ in $A_S \otimes {}_S(Ss)$. Injectivity of ι implies $u' = v'$. Thus $a \otimes us = a' \otimes vs$ in $A_S \otimes {}_S(Ss)$ and A_S is principally weakly flat. ∎

2.4 On weakly flat acts

Here we show that the second cell from the top in the leftmost column of Scheme 1 is the class of weakly flat right S-acts.

Proposition 2.4 *A right S-act A_S is weakly flat if and only if the corresponding φ is surjective for every pullback diagram $P(I, I, \iota, \iota, S)$, where I is a left ideal of S and $\iota : {}_SI \to {}_SS$ is a monomorphism of left S-acts.*

Proof. **Necessity.** Let A_S be weakly flat. Suppose that $a \otimes \iota(s) = a' \otimes \iota(t)$ for some $a, a' \in A_S, s, t \in I$ and a monomorphism $\iota : {}_SI \to {}_SS$ of left S-acts. Weak flatness of A_S implies that $a \otimes s = a' \otimes t$ in $A_S \otimes {}_SI$. Thus φ is surjective for the pullback diagram $P(I, I, \iota, \iota, S)$, because $a \otimes s = a' \otimes t$ and $a' \otimes t = a' \otimes t$ in $A_S \otimes {}_SI$ and $\iota(t) = \iota(t)$.

Sufficiency. Suppose that $a \otimes \iota(s) = a' \otimes \iota(t)$ in $A_S \otimes {}_SS$ for a left ideal I, $a, a' \in A_S$, $s, t \in I$ and a monomorphism $\iota : {}_SI \to {}_SS$. Surjectivity of φ for the diagram $P(I, I, \iota, \iota, S)$ implies that there exist $a'' \in A_S$, $u', v' \in I$ such that $\iota(u') = \iota(v')$, $a \otimes s = a'' \otimes u'$ and $a' \otimes t = a'' \otimes v'$ in $A_S \otimes {}_SI$. Injectivity of ι implies $u' = v'$. Thus $a \otimes s = a' \otimes t$ in $A_S \otimes {}_SI$ and A_S is weakly flat. ∎

2.5 On flat acts

The upper cell in the leftmost column of Scheme 1 is the class of all flat right S-acts.

Proposition 2.5 *A right S-act A_S is flat if and only if the corresponding φ is surjective for every pullback diagram $P(M, M, \iota, \iota, Q)$, where $\iota : {}_SM \to {}_SQ$ is a monomorphism of left S-acts.*

Proof. **Necessity.** Let A_S be flat. Suppose that $a \otimes \iota(m) = a' \otimes \iota(n)$ in $A_S \otimes {}_SQ$ for some $a, a' \in A_S$, $m, n \in {}_SM$ and a monomorphism $\iota : {}_SM \to {}_SQ$ of left S-acts. Flatness of A_S implies that $a \otimes m = a' \otimes n$ in $A_S \otimes {}_SM$. Thus φ is surjective for the pullback diagram $P(M, M, \iota, \iota, Q)$, because $a \otimes m = a' \otimes n$ and $a' \otimes n = a' \otimes n$ in $A_S \otimes {}_SM$ and $\iota(n) = \iota(n)$.

Sufficiency. Suppose that $a \otimes \iota(m) = a' \otimes \iota(n)$ in $A_S \otimes {}_SQ$ for $a, a' \in A_S$, $m, n \in {}_SM$ and a monomorphism $\iota : {}_SM \to {}_SQ$. Surjectivity of φ for the diagram $P(M, M, \iota, \iota, Q)$ implies that there exist $a'' \in A_S$, $m', n' \in M$ such that $\iota(m') = \iota(n')$, $a \otimes m = a'' \otimes m'$ and $a' \otimes n = a'' \otimes n'$ in $A_S \otimes {}_SM$. Injectivity of ι implies $m' = n'$. Thus $a \otimes m = a' \otimes n$ in $A_S \otimes {}_SM$ and A_S is flat. ∎

2.6 Principally weakly homoflat acts

Here we show that the two lowest cells in the second column of Scheme 1 define the same the class of right S-acts. We find an alternative description of a right S-act (cyclic right S-act, right Rees factor act) having the corresponding property and show that this class is a proper subclass of the class of all principally weakly flat right S-acts.

Proposition 2.6 *Let A_S be a right S-act. The following assertions are equivalent:*

1. *The corresponding φ is surjective for every pullback diagram $P(Ss, Ss, f, f, S)$, where $s \in S$.*
2. *The corresponding φ is surjective for every pullback diagram $P(S, S, f, f, S)$.*
3. $$(\forall a, a' \in A_S)(\forall t \in S)(at = a't \Rightarrow$$ $$(\exists a'' \in A_S)(\exists u, v \in S)(ut = vt \wedge a = a''u \wedge a' = a''v)).$$

Proof. 1. $\Rightarrow$ 2. is clear.

2. $\Rightarrow$ 3. Suppose that the corresponding φ is surjective for every pullback diagram $P(S, S, f, f, S)$. Let $at = a't$, $a, a' \in A_S$, $t \in S$. Consider the homomorphism $\rho_t : {}_SS \to {}_SS$. Then $\rho_t(1) = t$ and $a\rho_t(1) = a'\rho_t(1)$. By Lemma 1.3 we have $a \otimes \rho_t(1) = a' \otimes \rho_t(1)$ in $A_S \otimes {}_SS$. Since φ is surjective for the pullback diagram $P(S, S, \rho_t, \rho_t, S)$, there exist $a'' \in A_S$, $u, v \in S$ such that $\rho_t(u) = \rho_t(v)$, $a \otimes 1 = a'' \otimes u$ and $a' \otimes 1 = a'' \otimes v$ in $A_S \otimes {}_SS$. Using the definition of the homomorphism ρ_t and Lemma 1.3 we obtain $ut = vt$, $a = a''u$ and $a' = a''v$.

3. $\Rightarrow$ 1. Let $a \otimes f(xs) = a' \otimes f(ys)$ in $A_S \otimes {}_SS$ for $a, a' \in A_S, x, y, s \in S$ and a homomorphism $f : {}_S(Ss) \to {}_SS$. Let $t \in S$ be an element such that $f(s) = t$. Then we have $a \otimes xt = a' \otimes yt$ in $A_S \otimes {}_SS$ which by Lemma 1.3 means that $axt = a'yt$. By the assumption there exist $a'' \in A_S$, $u, v \in S$ such that $ut = vt$, $ax = a''u$ and $a'y = a''v$. Then

$$f(us) = uf(s) = ut = vt = vf(s) = f(vs),$$

$$a \otimes xs = ax \otimes s = a''u \otimes s = a'' \otimes us$$

and, analogously,

$$a' \otimes ys = a'' \otimes vs$$

in $A_S \otimes {}_S(Ss)$. Thus we have shown that φ is surjective for the pullback diagram $P(Ss, Ss, f, f, S)$. ∎

Definition 20 We say that a right S-act A_S is *principally weakly homoflat*, if the corresponding φ is surjective for every pullback diagram $P(Ss, Ss, f, f, S)$, $s \in S$.

Remark 1 The prefix 'homo' comes from the word 'homomorphism' and it indicates that we consider all homomorphisms instead of all monomorphisms (as it was in the case of principal weak flatness).

By Proposition 2.3 and Scheme 1 it is clear that principal weak homoflatness implies principal weak flatness.

Lemma 2.7 *Let ρ be a right congruence on a monoid S. The cyclic right S-act S/ρ is principally weakly homoflat if and only if*

$$(\forall x, y, t \in S)((xt)\rho(yt) \Rightarrow (\exists u, v \in S)(ut = vt \wedge x\rho u \wedge y\rho v)).$$

Proof. **Necessity.** Let S/ρ be principally weakly homoflat and let $(xt)\rho(yt)$ for some $x, y, t \in S$. Then we have $[x]_\rho t = [y]_\rho t$ in S/ρ. By Proposition 2.6 there exist $u', v', z \in S$ such that $u't = v't$, $[x]_\rho = [z]_\rho u'$ and $[y]_\rho = [z]_\rho v'$. Denoting $u = zu'$ and $v = zv'$ we have $ut = vt$, $x\rho u$ and $y\rho v$.

Sufficiency. Let $[x]_\rho t = [y]_\rho t$ for some $x, y, t \in S$. Then $(xt)\rho(yt)$ and applying the assumption we get $u, v \in S$ such that $ut = vt$, $x\rho u$ and $y\rho v$. Hence $[x]_\rho = [1]_\rho u$ and $[y]_\rho = [1]_\rho v$. Thus S/ρ is principally weakly homoflat by Proposition 2.6. ∎

Definition 21 We say that a right ideal K of a monoid S is *left annihilating* if

$$(\forall t \in S)(\forall x, y \in S \setminus K)(xt, yt \in K \Rightarrow xt = yt).$$

Observe that if K is a proper left annihilating right ideal then $xt \in K$ implies $xt = t$ for every $t \in K$ and every $x \in S \setminus K$.

Lemma 2.8 *Let K be a right ideal of a monoid S. The right Rees factor act S/K is principally weakly homoflat if and only if K is left stabilizing and left annihilating.*

Proof. **Necessity.** Let S/K be principally weakly homoflat. Then it is principally weakly flat and hence K is left stabilizing by Proposition 1.11. Suppose that $xt, yt \in K$ for some $t \in S$ and $x, y \in S \setminus K$. Then $(xt)\rho_K(yt)$. By Lemma 2.7 there exist $u, v \in S$ such that $ut = vt$, $x\rho_K u$ and $y\rho_K v$. Now $x, y \notin K$ yields $x = u$ and $y = v$ by the definition of ρ_K. Hence $xt = yt$ and so K is a left annihilating right ideal.

Sufficiency. Let K be a left stabilizing and left annihilating right ideal of S. We use Lemma 2.7 to check that S/K is principally weakly homoflat.

Let $(xt)\rho_K(yt)$ for some $x, y, t \in S$. If $xt = yt$ then we can take $u = x$ and $v = y$. So we may assume that $xt, yt \in K$. We have the following four cases to consider.

a) $x, y \in K$. Then we can take $u = v = x$.

b) $x \in K$, $y \notin K$. Since K is left stabilizing, we can find for $yt \in K$ an element $z \in K$ such that $zyt = yt$. So we can take $u = zy$ and $v = y$.

c) $x \notin K$, $y \in K$. This is analogous to the previous case.

d) $x, y \notin K$. Since K is a left annihilating right ideal, we have $xt = yt$, so $u = x$ and $v = y$ do the job. ∎

Corollary 2.9 *The one-element right S-act Θ_S is principally weakly homoflat.*

Proof. Recall that this S-act is isomorphic to the Rees factor act S/S. Clearly S is both left stabilizing and left annihilating right ideal. Hence this Rees factor act is principally weakly homoflat by Lemma 2.8. ∎

It turns out that principal weak flatness and principal weak homoflatness are different notions.

Example 1 (Flatness does not imply principal weak homoflatness.) Let $S = \{1, e, f, 0\}$ be a semilattice, where $ef = 0$. Consider a right ideal $K = eS = \{e, 0\}$ of S. Since e and 0 are idempotents, the Rees factor act S/K is principally weakly flat by Proposition 1.11 (and even flat, because S is commutative). On the other hand, it is not principally weakly homoflat. Indeed, $1, f \in S \setminus K$, $1e, fe \in K$, but $1e \neq fe$, so K is not left annihilating.

2.7 Weakly homoflat acts

Here we show that the second cell from the top in the second column of Scheme 1 is a proper subclass of the class of principally weakly homoflat acts and the class of weakly flat acts. We find an alternative description of a right S-act (cyclic right S-act, right Rees factor act) having the corresponding property.

Lemma 2.10 *Let A_S be a right S-act. The corresponding φ is surjective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S, if and only if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$, all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'' \in A_S$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$.*

Proof. **Necessity.** Suppose that φ is surjective for every pullback diagram $P(I, I, f, f, S)$. Let $f : {}_S(Ss \cup St) \to {}_SS$ be a homomorphism of left S-acts, $s, t \in S$. Suppose that $af(s) = a'f(t)$ for some $a, a' \in A_S$. This means that $a \otimes f(s) = a' \otimes f(t)$ in $A_S \otimes {}_SS$ by Lemma 1.3. By surjectivity of φ for the diagram $P(Ss \cup St, Ss \cup St, f, f, S)$ there exist $a'' \in A_S$, $u, v \in S, s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$.

Sufficiency. Suppose that the assumption holds. Consider a homomorphism $f : {}_SI \to {}_SS$. Let $a \otimes f(i) = a' \otimes f(j)$ in $A_S \otimes {}_SS$ for some $a, a' \in A_S, i, j \in I$. Then $af(i) = a'f(j)$ by Lemma 1.3. Set $J = Si \cup Sj \subseteq I$ and $h = f|_J : {}_S(Si \cup Sj) \to {}_SS$. Then $ah(i) = a'h(j)$ and by the assumption there exist $a'' \in A_S$, $u, v \in S$, $i', j' \in \{i, j\}$ such that $h(ui') = h(vj')$, $a \otimes i = a'' \otimes ui'$ and $a' \otimes j = a'' \otimes vj'$ in $A_S \otimes {}_SJ$. Now clearly $ui', vj' \in J$,

$$f(ui') = h(ui') = h(vj') = f(vj'),$$

and $J \subseteq I$ implies that $a \otimes i = a'' \otimes ui'$ and $a' \otimes j = a'' \otimes vj'$ in $A_S \otimes_S I$. Thus the corresponding φ is surjective for the pullback diagram $P(I, I, f, f, S)$. ∎

Definition 22 We say that a right S-act A_S is *weakly homoflat*, if the corresponding φ is surjective for every pullback diagram $P(I, I, f, f, S)$, where I is a left ideal of S.

By the definition weak homoflatness implies principal weak homoflatness and by Proposition 2.4 weak homoflatness implies weak flatness.

The next lemma gives one more description of weak homoflatness. Although its formulation is quite cumbersome its advantage, comparing with Lemma 2.10, is that the lengths of tossings involved do not exceed 3.

Lemma 2.11 *A right S-act A_S is weakly homoflat if and only if for all elements $s, t \in S$, all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$, all $a, a' \in A_S$, if $af(s) = a'f(t)$ then there exist $a'', a_1, a_2 \in A_S$, $u, v, p_1, p_2, q_1, q_2 \in S$ such that either $f(us) = f(vt)$ and*

$$\begin{aligned} a &= a_1p_1 & & \\ a_1q_1 &= a''u & p_1s &= q_1s \\ a' &= a_2p_2 & & \\ a_2q_2 &= a''v & p_2t &= q_2t, \end{aligned}$$

or $f(us) = f(vs)$ *and*

$$\begin{array}{rclrcl} a' &=& a_1p_1 & & & \\ a_1q_1 &=& a_2p_2 & p_1t &=& q_1t \\ a_2q_2 &=& a''v & p_2t &=& q_2s \\ a &=& a''u, & & & \end{array}$$

or $f(ut) = f(vt)$ *and*

$$\begin{array}{rclrcl} a &=& a_1p_1 & & & \\ a_1q_1 &=& a_2p_2 & p_1s &=& q_1s \\ a_2q_2 &=& a''u & p_2s &=& q_2t \\ a' &=& a''v. & & & \end{array}$$

Proof. **Necessity.** Let $af(s) = a'f(t)$ for $s, t \in S$, $a, a' \in A_S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. By Lemma 2.10 there exist $a'' \in A_S$, $u, v \in S$, $s', t' \in \{s, t\}$ such that $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$. Consequently $a \otimes s = a''u \otimes s'$ and $a' \otimes t = a''v \otimes t'$ in $A_S \otimes {}_S(Ss \cup St)$. By Lemma 1.2 there exist natural numbers n and m and elements $a_1, \ldots, a_{n-1}, a'_1, \ldots, a'_{m-1} \in A_S$, $z_1, \ldots, z_{n-1}, z'_1, \ldots, z'_{m-1} \in \{s, t\}$, $p_1, \ldots, p_n$, $q_1, \ldots, q_{n-1}$, $p'_1, \ldots, p'_m$, $q'_1, \ldots, q'_{m-1} \in S$ such that

$$\begin{array}{rclrcl} a &=& a_1p_1 & p_1s &=& q_1z_1 \\ a_1q_1 &=& a_2p_2 & p_2z_1 &=& q_2z_2 \\ & \cdots & & & \cdots & \\ a_{k-1}q_{k-1} &=& a_kp_k & p_kz_{k-1} &=& q_kz_k \\ a_kq_k &=& a_{k+1}p_{k+1} & p_{k+1}z_k &=& q_{k+1}z_{k+1} \\ & \cdots & & & \cdots & \\ a_{n-2}q_{n-2} &=& a_{n-1}p_{n-1} & p_{n-1}z_{n-2} &=& q_{n-1}z_{n-1} \\ a_{n-1}q_{n-1} &=& a''up_n & p_nz_{n-1} &=& s' \end{array}$$

and

$$\begin{array}{rclrcl} a' &=& a'_1p'_1 & p'_1t &=& q'_1z'_1 \\ a'_1q'_1 &=& a'_2p'_2 & p'_2z'_1 &=& q'_2z'_2 \\ & \cdots & & & \cdots & \\ a'_{l-1}q'_{l-1} &=& a'_lp'_l & p'_lz'_{l-1} &=& q'_lz'_l \\ a'_lq'_l &=& a'_{l+1}p'_{l+1} & p'_{l+1}z'_l &=& q'_{l+1}z'_{l+1} \\ & \cdots & & & \cdots & \\ a'_{m-2}q'_{m-2} &=& a'_{m-1}p'_{m-1} & p'_{m-1}z'_{m-2} &=& q'_{m-1}z'_{m-1} \\ a'_{m-1}q'_{m-1} &=& a''vp'_m & p'_mz'_{m-1} &=& t'. \end{array}$$

Denote $z_0 = s$, $z'_0 = t$, $z_n = s'$ and $z'_m = t'$. Consider the following three cases.

a) $s' = t$. Then there exists $k \in \{1, \ldots, n\}$ such that $z_{k-1} = s$ and $z_k = z_{k+1} = \ldots = z_n = s' = t$. Consequently

$$\begin{aligned}
a_k q_k f(t) &= a_{k+1} p_{k+1} f(t) = a_{k+1} f(p_{k+1} z_k) = a_{k+1} f(q_{k+1} z_{k+1}) \\
&= a_{k+1} q_{k+1} f(z_{k+1}) = a_{k+2} p_{k+2} f(z_{k+1}) = \ldots \\
&= a_{n-1} p_{n-1} f(z_{n-2}) = a_{n-1} f(p_{n-1} z_{n-2}) = a_{n-1} f(q_{n-1} z_{n-1}) \\
&= a_{n-1} q_{n-1} f(z_{n-1}) = a'' u p_n f(z_{n-1}) = a'' f(u p_n z_{n-1}) \\
&= a'' f(us') = a'' f(vt') = a'' f(v p'_m z'_{m-1}) = a'' v p'_m f(z'_{m-1}) \\
&= a'_{m-1} q'_{m-1} f(z'_{m-1}) = a'_{m-1} f(q'_{m-1} z'_{m-1}) \\
&= a'_{m-1} f(p'_{m-1} z'_{m-2}) = a'_{m-1} p'_{m-1} f(z'_{m-2}) = \ldots = a'_1 p'_1 f(t) \\
&= a' f(t).
\end{aligned}$$

We know that weak homoflatness implies principal weak homoflatness. Hence the equality $a_k q_k f(t) = a' f(t)$ implies by Proposition 2.6 that there exist $d_1 \in A_S$ and $x_1, x_2 \in S$ such that $a_k q_k = d_1 x_1$, $a' = d_1 x_2$ and $x_1 f(t) = x_2 f(t)$. Moreover,

$$as = a_1 p_1 s = a_1 q_1 z_1 = a_2 p_2 z_1 = \ldots = a_k p_k z_{k-1} = a_k p_k s$$

and so the equality $as = a_k p_k s$ implies the existence of $d_2 \in A_S$ and $y_1, y_2 \in S$ such that $a = d_2 y_1$, $a_k p_k = d_2 y_2$ and $y_1 s = y_2 s$. So we have $f(x_1 t) = f(x_2 t)$ and

$$\begin{aligned}
a &= d_2 y_1 & & \\
d_2 y_2 &= a_k p_k & y_1 s &= y_2 s \\
a_k q_k &= d_1 x_1 & p_k s &= q_k t \\
a' &= d_1 x_2. & &
\end{aligned}$$

b) $t' = s$. Then there exists $l \in \{1, \ldots, m\}$ such that $z'_{l-1} = t$ and $z'_l = z'_{l+1} = \ldots = z'_m = t' = s$. Discussing as in the case a) we get the 'dual' result.

c) $s' = s$ and $t' = t$. Then $as = a'' us$, $a't = a'' vt$ and $f(us) = f(vt)$. Applying Proposition 2.6 we get $a_1, a_2 \in A_S$, $p_1, p_2, q_1, q_2 \in S$ such that

$$\begin{aligned}
a &= a_1 p_1 & & \\
a_1 q_1 &= a'' u & p_1 s &= q_1 s \\
a' &= a_2 p_2 & & \\
a_2 q_2 &= a'' v & p_2 t &= q_2 t,
\end{aligned}$$

Sufficiency. We use Lemma 2.10 for proving weak homoflatness of A_S. Suppose that $af(s) = a'f(t)$ for some $a, a' \in A_S$, $s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_S S$. By assumption there exist elements

$a'', a_1, a_2 \in A_S$, $u, v, p_1, p_2, q_1, q_2 \in S$ such that one of the three possibilities holds.

If $f(us) = f(vt)$ and

$$\begin{array}{rclrcl} a &=& a_1 p_1 & & & \\ a_1 q_1 &=& a'' u & p_1 s &=& q_1 s \\ a' &=& a_2 p_2 & & & \\ a_2 q_2 &=& a'' v & p_2 t &=& q_2 t, \end{array}$$

then

$$a \otimes s = a_1 p_1 \otimes s = a_1 \otimes p_1 s = a_1 \otimes q_1 s = a_1 q_1 \otimes s = a'' u \otimes s = a'' \otimes us$$

and, similarly,

$$a' \otimes t = a'' \otimes vt$$

in $A_S \otimes {}_S(Ss \cup St)$.

If $f(us) = f(vs)$ and

$$\begin{array}{rclrcl} a' &=& a_1 p_1 & & & \\ a_1 q_1 &=& a_2 p_2 & p_1 t &=& q_1 t \\ a_2 q_2 &=& a'' v & p_2 t &=& q_2 s \\ a &=& a'' u, & & & \end{array}$$

then

$$a \otimes s = a'' u \otimes s = a'' \otimes us$$

and

$$\begin{aligned} a' \otimes t &= a_1 p_1 \otimes t = a_1 \otimes p_1 t = a_1 \otimes q_1 t = a_1 q_1 \otimes t = a_2 p_2 \otimes t = a_2 \otimes p_2 t \\ &= a_2 \otimes q_2 s = a_2 q_2 \otimes s = a'' v \otimes s = a'' \otimes vs \end{aligned}$$

in $A_S \otimes {}_S(Ss \cup St)$.

The third case is analogous to the second.

So in any case we have $f(us') = f(vt')$, $a \otimes s = a'' \otimes us'$ and $a' \otimes t = a'' \otimes vt'$ in $A_S \otimes {}_S(Ss \cup St)$ for some $a'' \in A_S$, $u, v \in S$ and $s', t' \in \{s, t\}$. ∎

For cyclic acts we have the following description of weak homoflatness.

Lemma 2.12 *Let ρ be a right congruence on a monoid S. The cyclic right S-act S/ρ is weakly homoflat if and only if for all elements $s, t \in S$ and* *all homomorphisms* $f : {}_S(Ss \cup St) \to {}_SS$, *if* $f(s)\rho f(t)$ *then there exist* $u, v, p_1, p_2, q_1, q_2 \in S$ *such that either* $f(us) = f(vt)$ *and*

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [u] & p_1 s &= q_1 s \\ [1] &= [p_2] & & \\ [q_2] &= [v] & p_2 t &= q_2 t, \end{aligned}$$

or $f(us) = f(vs)$ *and*

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 t &= q_1 t \\ [q_2] &= [v] & p_2 t &= q_2 s \\ [1] &= [u], & & \end{aligned}$$

or $f(ut) = f(vt)$ *and*

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 s &= q_1 s \\ [q_2] &= [u] & p_2 s &= q_2 t \\ [1] &= [v]. & & \end{aligned}$$

Proof. **Necessity.** Let S/ρ be weakly homoflat. Suppose that $f(s)\rho f(t)$ for some $s, t \in S$ and a homomomorphism $f : {}_S(Ss \cup St) \to {}_SS$. Then we have the equality $[1]f(s) = [1]f(t)$ in S/ρ. By Lemma 2.11 there exist $a'', a_1, a_2, u', v', p_1', p_2', q_1', q_2' \in S$ such that either $f(u's) = f(v't)$ and

$$\begin{aligned} [1] &= [a_1]p_1' & & \\ [a_1]q_1' &= [a'']u' & p_1' s &= q_1' s \\ [1] &= [a_2]p_2' & & \\ [a_2]q_2' &= [a'']v' & p_2' t &= q_2' t, \end{aligned}$$

or $f(u's) = f(v's)$ and

$$\begin{aligned} [1] &= [a_1]p_1' & & \\ [a_1]q_1' &= [a_2]p_2' & p_1' t &= q_1' t \\ [a_2]q_2' &= [a'']v' & p_2' t &= q_2' s \\ [1] &= [a'']u', & & \end{aligned}$$

or $f(u't) = f(v't)$ and

$$\begin{aligned} [1] &= [a_1]p_1' & & \\ [a_1]q_1' &= [a_2]p_2' & p_1' s &= q_1' s \\ [a_2]q_2' &= [a'']u' & p_2' s &= q_2' t \\ [1] &= [a'']v'. & & \end{aligned}$$

The claim follows if we use that ρ is a right congruence, f is a homomorphism and denote $p_1 = a_1 p_1'$, $q_1 = a_1 q_1'$, $p_2 = a_2 p_2'$, $q_2 = a_2 q_2'$, $u = a'' u'$ and $v = a'' v'$.

Sufficiency. We use Lemma 2.11 to prove that S/ρ is weakly homoflat. Suppose that $[a]f(s) = [a']f(t)$ for some $a, a', s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_S S$. Then $f(as)\rho f(a't)$. Denoting $h = f|_{Sas \cup Sa't}$ the restriction of f to the left ideal $Sas \cup Sa't$, we have $h(as)\rho h(a't)$. By assumption there exist elements $u, v, p_1, p_2, q_1, q_2 \in S$ such that either $h(uas) = h(va't)$ and

$$\begin{array}{rclrcl} [1] & = & [p_1] & & & \\ [q_1] & = & [u] & p_1 as & = & q_1 as \\ [1] & = & [p_2] & & & \\ [q_2] & = & [v] & p_2 a't & = & q_2 a't, \end{array}$$

or $h(uas) = h(vas)$ and

$$\begin{array}{rclrcl} [1] & = & [p_1] & & & \\ [q_1] & = & [p_2] & p_1 a't & = & q_1 a't \\ [q_2] & = & [v] & p_2 a't & = & q_2 as \\ [1] & = & [u], & & & \end{array}$$

or $h(ua't) = h(va't)$ and

$$\begin{array}{rclrcl} [1] & = & [p_1] & & & \\ [q_1] & = & [p_2] & p_1 as & = & q_1 as \\ [q_2] & = & [u] & p_2 as & = & q_2 a't \\ [1] & = & [v]. & & & \end{array}$$

Using that ρ is a right congruence and the definition of h, we obtain either $f(uas) = f(va't)$ and

$$\begin{array}{rclrcl} [a] & = & [1]p_1 a & & & \\ [1]q_1 a & = & [1]ua & p_1 as & = & q_1 as \\ [a'] & = & [1]p_2 a' & & & \\ [1]q_2 a' & = & [1]va' & p_2 a't & = & q_2 a't, \end{array}$$

or $f(uas) = f(vas)$ and

$$\begin{array}{rclrcl} [a'] & = & [1]p_1 a' & & & \\ [1]q_1 a' & = & [1]p_2 a' & p_1 a't & = & q_1 a't \\ [1]q_2 a & = & [1]va & p_2 a't & = & q_2 as \\ [a] & = & [1]ua, & & & \end{array}$$

or $f(ua't) = f(va't)$ and

$$\begin{array}{rclrcl} [a] &=& [1]p_1 a & & & \\ [1]q_1 a &=& [1]p_2 a & p_1 as &=& q_1 as \\ [1]q_2 a' &=& [1]ua' & p_2 as &=& q_2 a't \\ [a'] &=& [1]va'. & & & \end{array}$$

Hence S/ρ is weakly homoflat. ∎

Definition 23 We say that a right ideal K of a monoid S is *strongly left annihilating* if for all $s, t \in S \setminus K$ and for all homomorphisms $f : {}_S(Ss \cup St) \to {}_S S$

$$f(s), f(t) \in K \Rightarrow f(s) = f(t).$$

Every strongly left annihilating right ideal is left annihilating. Indeed, if $xt, yt \in K$ for $t \in S$ and $x, y \in S \setminus K$ then $\rho_t(x), \rho_t(y) \in K$. This implies, if K is strongly left annihilating, that $xt = yt$. Hence K is left annihilating.

It turns out that not all left annihilating ideals are strongly left annihilating.

Example 2 Let S be an annihilating chain of semigroup $S_1 = \{1\}$, a right zero semigroup $S_2 = \{s, t\}$, a left zero semigroup $S_3 = \{x, y\}$ and a semigroup $S_4 = \{0\}$ $(1 > 2 > 3 > 4)$. Consider a right ideal $K = \{x, y, 0\}$.

Suppose that $uz, vz \in K$, $z \in S$, $u, v \in S \setminus K$. Then $z \in K$ and $uz = vz = z$. Hence K is left annihilating.

Define a mapping $f : Ss \cup St \to S$ by

$$\begin{aligned} f(us) &= ux, \\ f(ut) &= uy, \end{aligned}$$

$u \in S$. It is straightforward to check that f is a homomorphism of left S-acts. Now $f(s), f(t) \in K$ but $f(s) \neq f(t)$. Thus K is not strongly left annihilating.

Lemma 2.13 *Let K be a right ideal of a monoid S. The right Rees factor act S/K is weakly homoflat if and only if S is right reversible, K is left stabilizing and K is strongly left annihilating.*

Proof. **Necessity.** Let S/K be weakly homoflat. Then S/K is weakly flat. Hence S is right reversible and K is left stabilizing by Proposition 1.11. Let us show that K is strongly left annihilating. If $K = S$ then the statement is obvious. Assume that $K \subset S$. Then $1 \notin K$ and hence the ρ_K-class of identity element is a singleton. Suppose that $f(s), f(t) \in K$ for some $s,t \in S \setminus K$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. This means that $f(s)\rho_K f(t)$. By Lemma 2.12 there are three possibilities.

a) There exist $u, v, p_1, p_2, q_1, q_2 \in S$ such that $f(us) = f(vt)$ and

$$\begin{aligned}
[1] &= [p_1] \\
[q_1] &= [u] \quad p_1 s = q_1 s \\
[1] &= [p_2] \\
[q_2] &= [v] \quad p_2 t = q_2 t.
\end{aligned}$$

Consequently $p_1 = 1 = p_2$. Now $s \notin K$ and $s = q_1 s$ imply that $q_1 \notin K$, therefore $q_1 = u$. Analogously $q_2 = v$. Thus

$$f(s) = f(q_1 s) = f(us) = f(vt) = f(q_2 t) = f(t).$$

b) There exist $u, v, p_1, p_2, q_1, q_2 \in S$ such that $f(us) = f(vs)$ and

$$\begin{aligned}
[1] &= [p_1] \\
[q_1] &= [p_2] \quad p_1 t = q_1 t \\
[q_2] &= [v] \quad p_2 t = q_2 s \\
[1] &= [u].
\end{aligned}$$

Then $p_1 = 1$ and $t = q_1 t$, which, together with $t \notin K$, implies that $q_1 \notin K$. Consequently $q_1 = p_2$ and $t = q_1 t = p_2 t = q_2 s$. This again implies that $q_2 \notin S$, hence $v = q_2$ and $t = vs$. Since $[1] = [u]$ here means that $u = 1$, we have got

$$f(s) = f(us) = f(vs) = f(t).$$

c) This case is similar to the previous one.

Sufficiency. Let S be right reversible, K left stabilizing and strongly left annihilating. We use Lemma 2.12 to show that $S/K = S/\rho_K$ is weakly homoflat. Suppose that $f(s)\rho_K f(t)$ for $s,t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. If $f(s) = f(t)$ then the elements u, v, p_1, p_2, q_1, q_2 we need can all be taken equal to 1. For the case $f(s), f(t) \in K$ let us consider the following four possibilities.

a) $s, t \in K$. Right reversibility of S implies that there exist $u', v' \in S$ such that $u's = v't$. Since K is left stabilizing, there exist $q_1, q_2 \in K$ such that $s = q_1 s$ and $t = q_2 t$. Take arbitrary $z \in K$ and denote $u = zu'$, $v = zv'$. Then we have

$$f(us) = f(zu's) = f(zv't) = f(vt)$$

and

$$\begin{aligned}
[1] &= [1] \\
[q_1] &= [u] \quad 1s = q_1 s \\
[1] &= [1] \\
[q_2] &= [v] \quad 1t = q_2 t.
\end{aligned}$$

Hence S/K is weakly homoflat.

b) $s \in K, t \notin K$. Right reversibility of S implies the existence of elements $u', v' \in S$ such that $u's = v't$. Since K is a left stabilizing right ideal, there exist $q_1, u \in K$ such that $s = q_1 s$ and $uf(t) = f(t)$. Take arbitrary $z \in K$ and denote $p_2 = zu'$, $q_2 = zv'$. Now we have

$$f(ut) = f(1t)$$

and

$$\begin{array}{rclrcl} [1] & = & [1] & & & \\ [q_1] & = & [p_2] & 1s & = & q_1 s \\ [q_2] & = & [u] & p_2 s & = & q_2 t \\ [1] & = & [1]. & & & \end{array}$$

Hence S/K is again weakly homoflat.

c) $s \notin K, t \in K$. This case is analogous to the previous one.

d) $s \notin K, t \notin K$. Since K is strongly left annihilating, $f(s) = f(t)$ and we are done as before. ∎

Lemma 2.14 *The following assertions are equivalent for a monoid S:*

1. *Θ_S satisfies condition* (P)*.*
2. *Θ_S is weakly homoflat.*
3. *Θ_S is weakly flat.*
4. *S is right reversible.*

Proof. 2. $\Rightarrow$ 3. is clear. 3. $\Rightarrow$ 4. and 4. $\Rightarrow$ 1. come from Corollary 1.12.

1. $\Rightarrow$ 2. By Corollary 1.12 Θ_S satisfies condition (P) if and only if S is right reversible. Clearly S is a left stabilizing and strongly left annihilating right ideal of S. Hence $\Theta_S \cong S/S$ is weakly homoflat by Lemma 2.13. ∎

Let us show that weak homoflatness and principal weak homoflatness are different notions.

Example 3 (Principal weak homoflatness does not imply weak homoflatness.) Let K be a right zero semigroup with two or more elements and let $S = K^1$ be a monoid obtained from K by external adjoining of identity. Clearly K is a right ideal of S and the Rees factor act S/K is not weakly flat (hence it cannot be weakly homoflat), because S is not right reversible. But S/K is principally weakly homoflat. To see this, let us use Lemma 2.8. Clearly K is a left stabilizing right ideal, so it remains to show that K is left annihilating. Suppose that $xt, yt \in K$ for some $x, y \in S \setminus K$ and $t \in S$. Since $|S \setminus K| = 1$, we immediately get that $x = y = 1$ and hence $xt = yt$. Thus K is left annihilating, too.

Moreover, flatness does not imply weak homoflatness, because otherwise flatness would imply principal weak homoflatness, which is not the case (see Example 1). However, the question whether weak homoflatness implies flatness remains open here.

2.8 On acts satisfying condition (P)

Here we see that the cells in the third column and the upper cell of the second column of Scheme 1 denote the same class of right S-acts — the class of all acts satisfying condition (P). We also give an example of a weakly homoflat right S-act which does not satisfy condition (P).

Proposition 2.15 *The following assertions are equivalent for a right S-act A_S:*

1. *The corresponding φ is surjective for every pullback diagram $P(M, N, f, g, Q)$.*
2. *The corresponding φ is surjective for every pullback diagram $P(M, M, f, g, Q)$.*
3. *The corresponding φ is surjective for every pullback diagram $P(I, I, f, g, S)$, where I is a left ideal of S.*
4. *The corresponding φ is surjective for every pullback diagram $P(Ss, Ss, f, g, S)$, $s \in S$.*
5. *The corresponding φ is surjective for every pullback diagram $P(S, S, f, g, S)$.*
6. *The corresponding φ is surjective for every pullback diagram $P(M, M, f, f, Q)$.*
7. *A_S satisfies condition* (P).

Proof. The proof of the equivalence of the conditions 1, 2, 3, 4, 5 and 7 follows directly from the proof of Lemma 2.2 of [2]. The implication 2. $\Rightarrow$ 6. is obvious. Let us show that 6. $\Rightarrow$ 7.

Assume that φ is surjective for every pullback diagram $P(M, M, f, f, Q)$. Suppose that $as = a's'$, $a, a' \in A_S$, $s, s' \in S$. Consider a mapping $f : {}_S(S_1 \bigsqcup S_2) \to {}_S S$ which is defined by

$$f(x_1) = xs,$$
$$f(x_2) = xs',$$

$x \in S$ (here we use the same notation as introduced after Lemma 1.15). Then

$$uf(x_1) = u(xs) = (ux)s = f((ux)_1) = f(ux_1),$$

and analogously we obtain $uf(x_2) = f(ux_2)$ for all $u, x \in S$. That means, f is a homomorphism of left S-acts. Moreover, $as = a's'$ means that $a \otimes f(1_1) = a' \otimes f(1_2)$ in $A_S \otimes {}_SS$. Using surjectivity of φ for the pullback diagram $P(S_1 \coprod S_2, S_1 \coprod S_2, f, f, S)$ we get $a'' \in A_S$, $u, v \in S$, $i, j \in \{1, 2\}$ such that $f(u_i) = f(v_j)$, $a \otimes 1_1 = a'' \otimes u_i$ and $a' \otimes 1_2 = a'' \otimes v_j$ in tensor product $A_S \otimes_S (S_1 \sqcup S_2)$. The equality $a \otimes 1_1 = a'' \otimes u_i$ means by Lemma 1.1 that there exist a natural number n, elements $a_1, \ldots, a_{n-1} \in A_S, b_1, \ldots, b_{n-1}$, $s_1, \ldots, s_n$, $t_1, \ldots, t_{n-1} \in S$ and indices $i_1, \ldots, i_{n-1} \in \{1, 2\}$ such that

$$\begin{array}{rclrcl} a & = & a_1 s_1 & s_1 1_1 & = & t_1 (b_1)_{i_1} \\ a_1 t_1 & = & a_2 s_2 & s_2 (b_1)_{i_1} & = & t_2 (b_2)_{i_2} \\ & \ldots & & & \ldots & \\ a_{n-1} t_{n-1} & = & a'' s_n & s_n (b_{n-1})_{i_{n-1}} & = & u_i. \end{array}$$

Since S_1 and S_2 are disjoint, the equality $s_1 1_1 = t_1 (b_1)_{i_1}$ implies that $i_1 = 1$. Analogously we get that $i_2 = i_3 = \ldots = i_{n-1} = 1$ and then also $i = 1$. Now all the equalities in the right-hand column above hold in S_1. Using that α_1 is a monomorphism of left S-acts we obtain

$$\begin{array}{rclrcl} a & = & a_1 s_1 & s_1 1 & = & t_1 b_1 \\ a_1 t_1 & = & a_2 s_2 & s_2 b_1 & = & t_2 b_2 \\ & \ldots & & & \ldots & \\ a_{n-1} t_{n-1} & = & a'' s_n & s_n b_{n-1} & = & u. \end{array}$$

which means that $a \otimes 1 = a'' \otimes u$ in $A_S \otimes {}_SS$. Analogously we get $j = 2$ and $a' \otimes 1 = a'' \otimes v$ in $A_S \otimes {}_SS$. By Lemma 1.3 this means that $a = a''u$ and $a' = a''v$. Finally, $f(u_1) = f(v_2)$ yields $us = vs'$ by the definition of f. ∎

Using this proposition we see that conditin (P) implies weak homoflatness. This implication is strict.

Example 4 (Weak homoflatness does not imply condition (P).) Let us consider a free monoid $K = s^*$ generated by one element s and let $e = s^0$ be the identity element of K. Let $S = K^1$ be a monoid obtained by external adjoining of identity 1 to K. Clearly K is a right ideal of S. The Rees factor S/K does not satisfy condition (P) (note that S/K is flat), because $K \neq S$ and $|K| > 1$. Let us show that S/K is weakly homoflat. Let $I \neq S$ be a left ideal of S. Then $I \subseteq K$. If k is the minimal nonnegative integer such that $s^k \in K$, then $I = Ss^k$. So all left ideals of S are principal. By the definitions this means that a right S-act is weakly homoflat if and only if it is principally weakly homoflat. Consequently, to show that S/K is weakly homoflat, it is sufficient to check that $xt, yt \in K$ with $t \in S$, $x, y \in S \setminus K$ implies $xt = yt$. But this is evident since $S \setminus K = \{1\}$.

2.9 Weakly pullback flat acts

Here we show that the three lowest cells in the rightmost column of Scheme 1 denote the same class of right S-acts. We find another description of a right S-act (cyclic right S-act, right Rees factor act) having the corresponding property and give examples showing that this class lies properly between the classes of all pullback flat right S-acts and all right S-acts satisfying condition (P).

First, let us introduce a generalization of condition (E):

(E′) $(\forall a \in A_S)(\forall s, s', z \in S)(as = as' \wedge sz = s'z \Rightarrow$
$(\exists a' \in A_S)(\exists u \in S)(a = a'u \wedge us = us'))$

and one more condition on a right S-act A_S:

(PF′) $(\forall a, a' \in A_S)(\forall s, s', t, t', z, w \in S)$
$(sz = tw \wedge s'z = t'w \wedge as = a's' \wedge at = a't' \Rightarrow$
$(\exists a'' \in A_S)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs' \wedge ut = vt'))$.

We also need the following lemma.

Lemma 2.16 ([2]) *If A_S satisfies condition* (P) *and $a \otimes m = a' \otimes m'$ in $A_S \otimes {}_SM$ for a left S-act ${}_SM$, $a, a' \in A_S$, $m, m' \in {}_SM$ then there exist $a'' \in A_S$ and $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $um = vm'$.*

Proposition 2.17 *The following assertions are equivalent for a right S-act A_S:*

1. *The corresponding φ is surjective and injective for every pullback diagram $P(I, I, f, g, S)$, where I is a left ideal of S.*
2. *The corresponding φ is surjective and injective for every pullback diagram $P(Ss, Ss, f, g, S)$, $s \in S$.*
3. *The corresponding φ is surjective and injective for every pullback diagram $P(S, S, f, g, S)$.*
4. *A_S satisfies condition* (PF′).
5. *A_S satisfies conditions* (P) *and* (E′).

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. is clear.

3. $\Rightarrow$ 4. Let the corresponding φ be surjective and injective for every pullback diagram $P(S,S,f,g,S)$. Then A_S satisfies condition (P) by Proposition 2.15. Let

$$\begin{aligned} sz &= tw, & as &= a's', \\ s'z &= t'w, & at &= a't', \end{aligned}$$

$a, a' \in A_S$, $s, s', t, t', z, w \in S$. Take the homomorphisms $\rho_z, \rho_w : {}_SS \to {}_SS$. Then $\rho_z(s) = \rho_w(t)$, $\rho_z(s') = \rho_w(t')$ and by Lemma 1.3 $a \otimes s = a' \otimes s'$ and $a \otimes t = a' \otimes t'$ in $A_S \otimes {}_SS$. This means that $\varphi(a \otimes (s,t)) = \varphi(a' \otimes (s',t'))$ for the φ corresponding to the diagram $P({}_SS, {}_SS, \rho_z, \rho_w, {}_SS)$. Using injectivity of φ we get the equality $a \otimes (s,t) = a' \otimes (s',t')$ in $A_S \otimes {}_SP$, where

$${}_SP = \{(u,v) \in S \times S \mid \rho_z(u) = \rho_w(v)\} = \{(u,v) \in S \times S \mid uz = vw\}.$$

Since A_S satisfies conditon (P), by Lemma 2.16 there exist $a'' \in A_S$, $u, v \in S$ such that $a = a''u$, $a' = a''v$ and $u(s,t) = v(s',t')$. But then $us = vs'$ and $ut = vt'$.

4. $\Rightarrow$ 5. Condition (P) follows by taking $t = s$, $t' = s'$ and $z = w = 1$ in condition (PF$'$). Let us show that condition (E$'$) holds. Suppose that $as = as'$, $sz = s'z$, $a \in A_S$, $s, s', z \in S$. The equalities

$$\begin{aligned} sz &= 1sz, & as &= as', \\ s'z &= 1sz, & a1 &= a1, \end{aligned}$$

imply the existence of $a'' \in A_S$ and $u, v \in S$ such that $us = vs'$, $u1 = v1$ and $a = a''u$.

5. $\Rightarrow$ 1. Since A_S satisfies condition (P), the corresponding φ is surjective for every pullback diagram $P(I,I,f,g,S)$ by Proposition 2.15. Let us show that φ is also injective for every pullback diagram $P(I,I,f,g,S)$. Take such a diagram and suppose that there exist $i, i', j, j' \in I$ and $a, a' \in A_S$ such that

$$\begin{aligned} f(i) &= g(j), & a \otimes i &= a' \otimes i' \text{ in } A_S \otimes {}_SI \\ f(i') &= g(j'), & a \otimes j &= a' \otimes j' \text{ in } A_S \otimes {}_SI. \end{aligned}$$

Then the equalities $a \otimes i = a' \otimes i'$ and $a \otimes j = a' \otimes j'$ hold also in $A_S \otimes {}_SS$ and therefore $ai = a'i'$ and $aj = a'j'$ by Lemma 1.3. Using condition (P) we get from the equality $ai = a'i'$ that there exist $u_1, v_1 \in S$ and $b \in A_S$ such that $a = bu_1$, $a' = bv_1$ and $u_1 i = v_1 i'$. Therefore $bu_1 j = aj = a'j' = bv_1 j'$. Once more applying condition (P) we get from the equality $bu_1 j = bv_1 j'$ that there exist $u_2, v_2 \in S$ and $d \in A_S$ such that $b = du_2 = dv_2$ and $u_2 u_1 j = v_2 v_1 j'$. So

$$\begin{aligned} u_2 u_1 g(j) &= g(u_2 u_1 j) = g(v_2 v_1 j') = v_2 v_1 g(j') = v_2 v_1 f(i') = v_2 f(v_1 i') \\ &= v_2 f(u_1 i) = v_2 u_1 f(i) = v_2 u_1 g(j). \end{aligned}$$

The equalities $du_2 = dv_2$ and $u_2(u_1g(j)) = v_2(u_1g(j))$ yield by condition (E$'$) the existence of $w \in S$ and $a'' \in A_S$ such that $d = a''w$ and $wu_2 = wv_2$. Hence

$$\begin{aligned} a \otimes (i,j) &= bu_1 \otimes (i,j) = du_2u_1 \otimes (i,j) = a''wu_2u_1 \otimes (i,j) \\ &= a'' \otimes (wu_2u_1i, wu_2u_1j) = a'' \otimes (wv_2v_1i', wv_2v_1j') \\ &= a''wv_2v_1 \otimes (i',j') = dv_2v_1 \otimes (i',j') = bv_1 \otimes (i',j') \\ &= a' \otimes (i',j') \end{aligned}$$

in tensor product $A_S \otimes {}_SP$, where

$${}_SP = \{(m,n) \in {}_SI \times {}_SI \mid f(m) = g(n)\}.$$

Thus φ is injective for the pullback diagram $P(I,I,f,g,S)$. ∎

Equivalence of conditions 3, 4 and 5 was proved in [27].

Definition 24 We say that a right S-act A_S is *weakly pullback flat*, if the corresponding φ is surjective for every pullback diagram $P(I,I,f,g,S)$, where I is a left ideal of S.

Since condition (E) implies condition (E$'$), it follows from Theorem 1.9 and Proposition 2.17 that pullback flatness implies weak pullback flatness.

Lemma 2.18 *Let ρ be a right congruence on a monoid S. The cyclic right S-act S/ρ is weakly pullback flat if and only if it satisfies condition* (P) *and*

$$(\forall s,s',z \in S)(s\rho s' \wedge sz = s'z \Rightarrow (\exists u \in S)(u\rho 1 \wedge us = us')).$$

Proof. **Necessity.** Let S/ρ be weakly pullback flat. By Proposition 2.17 S/ρ satisfies conditions (P) and (E$'$). Now suppose that $s\rho s'$ and $sz = s'z$ for some $s,s',z \in S$. Then $[1]_\rho s = [1]_\rho s'$. Condition (E$'$) implies the existence of x,u' such that $[1]_\rho = [x]_\rho u'$ and $u's = u's'$. Denoting $u = xu'$ we have $u\rho 1$ and $us = us'$.

Sufficiency. By Proposition 2.17 it is sufficient to show that S/ρ satisfies condition (E$'$). Suppose that $[x]_\rho s = [x]_\rho s'$ and $sz = s'z$ for some $x,s,s',z \in S$. Then $(xs)\rho(xs')$ and $(xs)z = (xs')z$. Hence by the assumption there exists $u \in S$ such that $u\rho 1$ and $uxs = uxs'$. Consequently, $[x]_\rho = [1]_\rho ux$ and $(ux)s = (ux)s'$. ∎

Definition 25 Let S be a monoid and $P \subseteq S$ its submonoid. We shall say that the submonoid P is *weakly left collapsible* if

$$(\forall s,s' \in P)(\forall z \in S)(sz = s'z \Rightarrow (\exists u \in P)(us = us')).$$

Every left collapsible submonoid is weakly left collapsible. The converse is not true: take a nontrivial group as an example.

Lemma 2.19 *Let K be a right ideal of a monoid S. The right Rees factor act S/K is weakly pullback flat if and only if $|K| = 1$ or $K = S$ is right reversible and weakly left collapsible.*

Proof. **Necessity.** Since S/K is weakly pullback flat, it satisfies condition (P). By Proposition 1.11 either $|K| = 1$ or $K = S$ is right reversible. If $K = S$ then $s\rho_K s'$ for all $s, s' \in S$ and hence $sz = s'z$, $s, s', z \in S$ implies by Lemma 2.18 the existence of $u \in S$ such that $us = us'$. This means that S is weakly left collapsible.

Sufficiency. If $|K| = 1$ then $S/K \cong S$ is free and hence also weakly pullback flat. Suppose that $K = S$ is right reversible and weakly left collapsible. Then $S/K \cong \Theta_S$. By Corollary 1.12 S/K satisfies condition (P). By Lemma 2.18 it remains to show that $sz = s'z$, $s, s', z \in S$ implies the existence of $u \in S$ such that $us = us'$, but this is exactly weak left collapsibility of S. ∎

For the one-element right S-act Lemma 2.19 yields the following result.

Corollary 2.20 *The one-element right S-act Θ_S is weakly pullback flat if and only if S is right reversible and weakly left collapsible.*

Example 5 (Weak pullback flatness does not imply pullback flatness.) Let S be a nontrivial group. Then it is right reversible and weakly left collapsible and hence the one-element right S-act Θ_S is weakly pullback flat. But it cannot be pullback flat, because it does not satisfy condition (E). Moreover, this means that condition (E$'$) does not imply (E), because otherwise weak pullback flatness would imply pullback flatness.

To show that condition (P) and weak pullback flatness are different concepts we need a lemma.

Lemma 2.21 ([27]) *Let S be a right collapsible monoid. Then every weakly pullback flat right S-act A_S is pullback flat.*

Proof. Let A_S be weakly pullback flat. It is sufficient to show that A_S satisfies condition (E). Suppose that $as = as'$, $a \in A_S$, $s, s' \in S$. By the right collapsibility of S there exist $z \in S$ such that $sz = s'z$. Since A_S satisfies condition (E$'$) by Proposition 2.17, there exist $a' \in A_S$, $u \in S$ such that $a = a'u$, and $us = us'$. But this means that A_S satisfies condition (E) and hence it is pullback flat. ∎

Example 6 (Condition (P) does not imply weak pullback flatness.) Let $T = s^*$ be a monogenic free monoid generated by an element s and let S be a monoid obtained from T by external adjoining of zero 0. Since S is right collapsible, a right S-act is weakly pullback flat if and only if it is pullback flat. Let us consider the monocyclic right S-act $S/\rho(1, s)$, where $\rho(1, s)$ is the smallest right congruence identifying the elements 1 and s. By [1], Proposition 2.10, this act satisfies condition (P) and it is pullback flat if and only if s is an aperiodic element. Since s is not an aperiodic element, $S/\rho(1, s)$ cannot be pullback flat and hence it cannot be weakly pullback flat either.

2.10 On pullback flat acts

Finally we note that the two upper cells in the rightmost column of Scheme 1 denote the class of pullback flat acts.

From [2] we have the following condition on a right S-act A_S:

$$\text{(PF)}\quad (\forall a, a' \in A_S)(\forall s, s', t, t' \in S)(as = a's' \wedge at = a't' \Rightarrow$$
$$(\exists a'' \in A_S)(\exists u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vs' \wedge ut = vt')).$$

It is easy to see that condition (PF) implies condition (PF$'$).

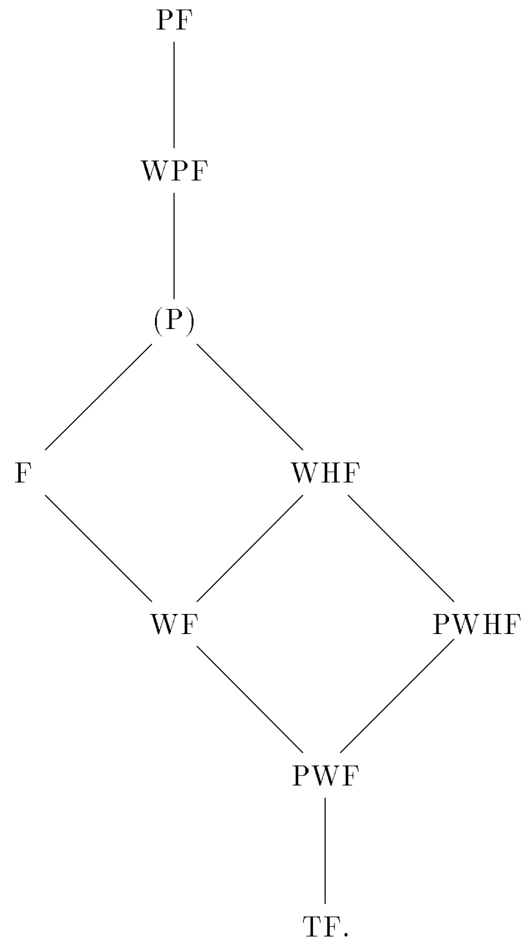
Proposition 2.22 *The following assertions are equivalent for a right S-act A_S:*

1. *The corresponding φ is surjective and injective for every pullback diagram $P(M, N, f, g, Q)$.*
2. *The corresponding φ is surjective and injective for every pullback diagram $P(M, M, f, g, Q)$.*
3. *A_S satisfies conditions* (P) *and* (E).
4. *A_S satisfies condition* (PF).

Proof. The equivalence of conditions 1 and 3 was proved in [32]. The equivalence of all four conditions follows from Theorems 2.3 and 2.4 of [2]. It even follows that these conditions are equivalent to the surjectivity and injectivity of the corresponding φ for the pullback diagram $P(S, S, c_\theta, c_\theta, \Theta)$, where Θ is the one-element left S-act and $c_\theta : S \to \Theta$ is the constant mapping. ∎

2.11 Conclusion

After showing which properties of acts under consideration are the same and which are different, we have come to the following scheme for the relationships between them:



Scheme 2

Here the abbreviations stand for the following properties of S-acts:

PF	— pullback flatness
WPF	— weak pullback flatness
(P)	— condition (P)
F	— flatness
WF	— weak flatness
WHF	— weak homoflatness
PWHF	— principal weak homoflatness
PWF	— principal weak flatness
TF	— torsion freeness.

A line between two properties means that the property at the higher end of the line implies the property at the lower end of the line and the converse is not true.

Scheme 2 is not the only possible one to depict flatness properties lying between strong flatness and torsion freeness. There are different requirements (e.g. only injectivity of φ for some kind of pullback diagrams) that are not considered here at all. Moreover, there are well-known flatness properties (condition (E), equalizer flatness) that do not appear in Scheme 2. This leaves open the possibility of composing a scheme which includes all so far studied flatness properties in terms of preserving pullbacks.

3 ON HOMOLOGICAL CLASSIFICATION

In this section we consider the homological classification. That means we consider the questions like "When all right acts with a property X have another property Y?" and "When all right acts have property Y?". We seek for answers referring only to internal properties of a monoid over which these acts are considered. The properties we consider here are torsion freeness, principal weak flatness, principal weak homoflatness, weak homoflatness, condition (P), weak pullback flatness, pullback flatness, projectivity and freeness. We have added projectivity and freeness to make the picture more complete, and we almost do not look at the questions related to flatness and weak flatness.

There are several possible levels for considering problems of homological classification. We here try to classify monoids by properties of right Rees factor acts, cyclic right acts and arbitrary right acts over them. In some cases, where the answer to a homological classification problem is not known for the general situation, we try to clarify the situation in the (simpler) class of idempotent monoids.

3.1 Principal weak flatness

Here we give a characterization of those monoids over which all torsion free right S-acts are principally weakly flat.

First we say some words on almost regular monoids.

Definition 26 We say that an element s of a monoid S is *left almost regular* if there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ s_m c_m &= s_{m-1} r_m \\ s &= s_m r s. \end{aligned}$$

A monoid S is *left almost regular* if all its elements are left almost regular.

Almost regular monoids were introduced in [9] where it was proved that all divisible right S-acts are principally weakly injective if and only if S is **right** almost regular. (In [9] those monoids were called simply almost regular and the indices were used in a slightly different way.) The definition of a left almost regular monoid is obtained by dualization of the definition of a right almost regular monoid in the sense that 'left cancellable' is replaced by 'right cancellable' and the order of factors in products is reversed.

Later it turns out that if we require the principal weak flatness of all torsion free right acts then we get the class of **left** almost regular monoids. This result may be considered as further evidence of some kind of duality of the properties of acts grouped around projectivity and injectivity.

It is easy to see that if every element of S is either regular or right cancellable then S is left almost regular — a fact which holds also for left PP monoids. PP monoids were first investigated by Kilp [14], and the term was introduced by Fountain in [8] .

Definition 27 A monoid S is called *left PP monoid* if for every $s \in S$ there exists an idempotent $e \in S$ such that $es = s$ and for all $u, v \in S$, $us = vs$ implies $ue = ve$.

It turns out that the class of left almost regular monoids lies properly between the class of left PP monoids and the class of monoids, every element of which is either regular or right cancellable.

Proposition 3.1 ([26]) *Every left almost regular monoid is a left PP monoid.*

Proof. Let S be a left almost regular monoid and $s \in S$. Then there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{array}{rcl} s_1c_1 & = & sr_1 \\ s_2c_2 & = & s_1r_2 \\ & \cdots & \\ s_mc_m & = & s_{m-1}r_m \\ s & = & s_mrs. \end{array}$$

From the first and the last equality we get

$$s_1c_1 = sr_1 = s_mrsr_1 = s_mrs_1c_1.$$

Since c_1 is right cancellable, $s_1 = s_mrs_1$. Using this, from the second equality we get

$$s_2c_2 = s_1r_2 = s_mrs_1r_2 = s_mrs_2c_2,$$

which implies $s_2 = s_mrs_2$. Continuing in this manner we finally obtain $s_m = s_mrs_m$, and hence $e = s_mr$ is an idempotent such that $es = s$.

Now let $us = vs$. Then

$$us_1c_1 = usr_1 = vsr_1 = vs_1c_1$$

and hence $us_1 = vs_1$. Further,

$$us_2c_2 = us_1r_2 = vs_1r_2 = vs_2c_2$$

implies $us_2 = vs_2$ and continuing in this manner we get $us_m = vs_m$. Hence $us_mr = vs_mr$ and thus S is a left PP monoid. ∎

The following example shows that the class of left almost regular monoids is a proper subclass of the class of left PP monoids.

Example 7 Let $K = s^*$ be a free monoid with a generating element s and identity element e. Let $S = K^1$, i.e. let S be obtained from K by the external adjoining of the identity element 1 (see Example 4). Then S is commutative and one can see that it is a PP monoid by direct checking or by applying a result of [14], which says that a commutative monoid is a PP monoid if and only if it is a semilattice of cancellative monoids. But S cannot be an almost regular monoid, because the only cancellable element is 1 and hence the almost regularity of s would mean the existence of the elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m \in S$ such that

$$s = s_mrs = s_{m-1}r_mrs = \ldots = s_1r_2\ldots r_mrs = sr_1r_2\ldots r_mrs,$$

which contradicts the non-regularity of s.

There exist left almost regular monoids, which have elements that are neither regular nor right cancellable.

Example 8 Let $S = \langle e, s, c \mid e^2 = e, es = se = ec = ce = s, sc = cs = s^2\rangle \cup \{1\}$. It is not difficult to see that S is a commutative monoid consisting of the elements of the form $1, e, s^k\,(k \in \mathbb{N})$ and $c^k\,(k \in \mathbb{N})$. The elements of the form c^k and 1 are the only cancellable elements, whereas 1 and e are the only regular elements. But since $ec^k = s^k$ and $s^k = es^k$, the elements s^k are also almost regular, although they are neither regular nor cancellable. Thus S is almost regular.

We shall now give a construction of a right ideal of a monoid S with some specific properties.

Take an element $s \in S$. Let $L(s)$ be the subset of S consisting of all elements $t \in S$ for which there exist elements $r_1, \ldots, r_m$, $s_1, \ldots, s_{m-1} \in S$

and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ t c_m &= s_{m-1} r_m. \end{aligned}$$

Since $s1 = s1$, we see that $s \in L(s)$ and so $L(s)$ is nonempty. Let $K_{TF}(s)$ be the right ideal of S generated by the set $L(s)$, i.e.

$$K_{TF}(s) = \bigcup_{t \in L(s)} tS.$$

Lemma 3.2 *For every $s \in S$ the right ideal $K_{TF}(s)$ is the smallest right ideal J containing the element s such that the right Rees factor act S/J is torsion free.*

Proof. As we saw, $K_{TF}(s)$ is a right ideal containing s. Let us show that $S/K_{TF}(s)$ is torsion free. Suppose that $s'c \in K_{TF}(s)$ for $s' \in S$ and right cancellable $c \in S$. Then $s'c \in tS$ for some $t \in L(s)$, hence there exist $r_1, \ldots, r_m, r_{m+1}, s_1, \ldots, s_{m-1} \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ t c_m &= s_{m-1} r_m \\ s'c &= t r_{m+1}. \end{aligned}$$

This means $s' \in L(s)$ and so $s' \in K_{TF}(s)$. Thus $S/K_{TF}(s)$ is torsion free by Proposition 1.11.

Now suppose that K is a right ideal of S containing s such that S/K is torsion free. We want to show that $K_{TF}(s) \subseteq K$. Take $tz \in K_{TF}(s)$, $t \in L(s)$, $z \in S$. Then there exist $r_1, \ldots, r_m, s_1, \ldots, s_{m-1} \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ t c_m &= s_{m-1} r_m. \end{aligned}$$

Now $s \in K$ implies $s_1 c_1 \in K$. Since S/K is torsion free, $s_1 \in K$ by Proposition 1.11. Analogously $s_2, \ldots, s_{m-1}, t \in K$ and hence $tz \in K$. ∎

If $x, y \in S$ then denote

$$K_{TF}(x, y) = K_{TF}(x) \cup K_{TF}(y).$$

As in the proof of Lemma 3.2 one can see that $K_{TF}(x, y)$ is the smallest right ideal containing elements x and y such that the right Rees factor act by it is torsion free.

The next proposition is the reason why we have been discussing almost regular monoids.

Proposition 3.3 ([26]) *The following assertions are equivalent for a monoid S:*

1. *All torsion free right S-acts are principally weakly flat.*

2. *All cyclic torsion free right S-acts are principally weakly flat.*

3. *All torsion free right Rees factor acts of S are principally weakly flat.*

4. *S is a left almost regular monoid.*

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. is clear.

3. $\Rightarrow$ 4. Suppose all torsion free right Rees factor acts are principally weakly flat. Take an element $s \in S$ and the right ideal $K_{TF}(s)$. By Lemma 3.2 the right Rees factor act $S/K_{TF}(s)$ is torsion free. By assumption $S/K_{TF}(s)$ must be principally weakly flat. Hence by Proposition 1.11 for $s \in K_{TF}(s)$ we can find $tr \in K_{TF}(s)$, where $t \in L(s), r \in S$, such that $trs = s$. The last equality together with the fact that $t \in L(s)$ yields that s is left almost regular.

4. $\Rightarrow$ 1. Let S be left almost regular. Assume that A_S is a torsion free act. Let $as = a's$, for $a, a' \in A_S, s \in S$. Since s is left almost regular, there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_m \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{array}{rcl} s_1c_1 & = & sr_1 \\ s_2c_2 & = & s_1r_2 \\ & \cdots & \\ s_mc_m & = & s_{m-1}r_m \\ s & = & s_mrs. \end{array}$$

Using the first equality we get

$$as_1c_1 = asr_1 = a'sr_1 = a's_1c_1.$$

Since A_S is torsion free, we get $as_1 = a's_1$. Analogously we obtain the equalities $as_2 = a's_2, \ldots, as_m = a's_m$. Then, clearly, $as_m r = a's_m r$. Now we have

$$a \otimes s = a \otimes s_m rs = as_m r \otimes s = a's_m r \otimes s = a' \otimes s_m rs = a' \otimes s$$

in the tensor product $A_S \otimes_S (Ss)$ which means that A_S is principally weakly flat by Lemma 1.6. ∎

To prove the following corollary we need a lemma.

Lemma 3.4 ([23], [20]) *The following assertions are equivalent for a monoid S:*

1. *All right S-acts are torsion free.*
2. *All cyclic right S-acts are torsion free.*
3. *All right Rees factor acts of S are torsion free.*
4. *Every right cancellable element of S is right invertible.*

Equivalence of conditions 1, 2 and 4 was proved in [23], and the equivalence of 3 and 4 is in [20].

We can now give a new proof of the following result.

Corollary 3.5 ([15]) *The following assertions are equivalent for a monoid S:*

1. *All right S-acts are principally weakly flat.*
2. *All cyclic right S-acts are principally weakly flat.*
3. *All right Rees factor acts of S are principally weakly flat.*
4. *S is a regular monoid.*

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. is clear.

3. $\Rightarrow$ 4. If all right Rees factor acts are principally weakly flat then all right Rees factor acts are torsion free and all torsion free Rees factor acts are principally weakly flat. Hence by Lemma 3.4 every right cancellable element of S is right invertible and by Proposition 3.3 S is left almost

regular. Take $s \in S$. Then there exist elements $r, r_1, \ldots, r_m$, $s_1, \ldots, s_m \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= s r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ s_m c_m &= s_{m-1} r_m \\ s &= s_m r s. \end{aligned}$$

Multiplying the corresponding equality by the right inverse c_i^{-1} of c_i, $i \in \{1, \ldots, m\}$,we get

$$\begin{aligned} s_1 &= s r_1 c_1^{-1} \\ s_2 &= s_1 r_2 c_2^{-1} \\ &\cdots \\ s_m &= s_{m-1} r_m c_m^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} s &= s_m r s = s_{m-1} r_m c_m^{-1} r s = s_{m-2} r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s = \ldots \\ &= s r_1 c_1^{-1} \ldots r_{m-1} c_{m-1}^{-1} r_m c_m^{-1} r s, \end{aligned}$$

i.e. s is regular.

4. $\Rightarrow$ 1. If S is regular then it is left almost regular and every right cancellable element is right invertible. Hence all right Rees factor acts are torsion free by Lemma 3.4 and all torsion free right Rees factor acts are principally weakly flat by Proposition 3.3. Thus all right Rees factor acts are principally weakly flat. ∎

3.2 Principal weak homoflatness

In this subsection we characterize monoids over which all (all torsion free, all principally weakly flat) right Rees factor acts are principally weakly homoflat, monoids over which all torsion free right acts are principally weakly homoflat and idempotent monoids over which all right Rees factor acts are principally weakly homoflat. We see that if all torsion free right S-acts are weakly homoflat then all torsion free right S-acts are weakly pullback flat and if all right S-acts are principally weakly homoflat then all right S-acts are weakly pullback flat.

We start with Rees factor acts.

Proposition 3.6 *The following assertions are equivalent for a monoid S:*

1. *All principally weakly flat right Rees factor acts of S are principally weakly homoflat.*

2. Every left stabilizing right ideal of S is left annihilating.

3.

$$\begin{array}{l}(\forall t, x, y, x_0, y_0, x_1, y_1, x_2, y_2, \ldots \in S)\\((x_0 = xt \wedge (\forall i \in \mathbb{N}_0)(x_{i+1}x_i = x_i)\wedge\\y_0 = yt \wedge (\forall i \in \mathbb{N}_0)(y_{i+1}y_i = y_i) \wedge x_0 \neq y_0) \Rightarrow\\(\exists p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\})(\exists z \in S)(x = pz \vee y = pz)).\end{array}$$

Proof. 1. $\Rightarrow$ 2. Let K be a left stabilizing right ideal of S. Then the Rees factor act S/K is principally weakly flat by Proposition 1.11. By assumption S/K is principally weakly homoflat. Hence K is a left annihilating right ideal by Lemma 2.8.

2. $\Rightarrow$ 3. Suppose that

$$x_0 = xt \wedge (\forall i \in \mathbb{N}_0)(x_{i+1}x_i = x_i) \wedge y_0 = yt \wedge (\forall i \in \mathbb{N}_0)(y_{i+1}y_i = y_i)$$

for some $t, x, y, x_0, y_0, x_1, y_1, x_2, y_2, \ldots \in S$, $x_0 \neq y_0$. Consider a right ideal

$$K = \left(\bigcup_{i\in\mathbb{N}_0} x_iS\right) \bigcup \left(\bigcup_{i\in\mathbb{N}_0} y_iS\right).$$

For every $k \in K$ there exists $l \in K$ such that $lk = k$, that is K is left stabilizing. Hence S/K is principally weakly flat by Proposition 1.11. By assumption K is left annihilating. Since $xt, yt \in K$ and $xt \neq yt$, either $x \in K$ or $y \in K$. Thus either $x = pz$ or $y = pz$ for some $p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\}$ and $z \in S$.

3. $\Rightarrow$ 1. Let S/K be a principally weakly flat right Rees factor act. Then K is left stabilizing by Proposition 1.11. We have to show that K is left annihilating. Suppose that $x_0 = xt \in K, y_0 = yt \in K$ for some $x, y \in S \setminus K$ and $t \in S$. Suppose that $x_0 \neq y_0$. Since K is left stabilizing, there exist $x_1, y_1, x_2, y_2, \ldots \in K$ such that $x_{i+1}x_i = x_i$ and $y_{i+1}y_i = y_i$ for every nonnegative integer i. By assumption there exist $p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\}$ and $z \in S$ such that either $x = pz$ or $y = pz$. Hence either $x \in K$ or $y \in K$, a contradiction. So we must have the equality $xt = yt$, that means K is left annihilating. ∎

Example 9 Recall examples 4 and 7. There we had $S = K^1$ where $K = s^*$ was a free monogenic monoid. Clearly S is a left annihilating right ideal of S. It was shown in Example 4 that every proper right ideal of S has form s^kS for some nonnegative integer k (note that S is commutative). If $k > 0$ then s^kS is not left stabilizing. If $k = 0$ then $s^0S = K$ is left annihilating as shown in Example 4. Thus all principally weakly flat right Rees factor acts of S are principally weakly homoflat by Proposition 3.6. But in Example 7 we saw that S is not almost regular and hence not all torsion free right Rees factor acts are principally weakly homoflat by Proposition 3.3.

Corollary 3.7 *All torsion free right Rees factor acts of a monoid S are principally weakly homoflat if and only if S is left almost regular and*

$$(\forall x, y, t \in S)(xt \neq yt \Rightarrow (x \in K_{TF}(xt, yt) \vee y \in K_{TF}(xt, yt))).$$

Proof. **Necessity.** If all torsion free right Rees factor acts are principally weakly homoflat then all torsion free right Rees factor acts are principally weakly flat and hence S is left almost regular by Proposition 3.3. Suppose $xt \neq yt$, $x, y, t \in S$. Let $K = K_{TF}(xt, yt)$. Then S/K is torsion free and by assumption it is principally weakly homoflat. Since $xt, yt \in K$ and K is left annihilating, either x or y is in K.

Sufficiency. Suppose S/K is a torsion free right Rees factor act. Left almost regularity of S implies that S/K is principally weakly flat by Proposition 3.3 and hence K is left stabilizing by Proposition 1.11. It remains to show that K is left annihilating and then apply Lemma 2.8. Suppose that $xt, yt \in K$ for some $x, y \in S \setminus K$. Then $K_{TF}(xt, yt) \subseteq K$ because $K_{TF}(xt, yt)$ is the smallest right ideal containing xt and yt such that the Rees factor by it is torsion free. If $xt \neq yt$ then by assumption either $x \in K$ or $y \in K$, a contradiction. Hence $xt = yt$. ∎

Corollary 3.8 *All right Rees factor acts of a monoid S are principally weakly homoflat if and only if S is regular and*

$$(\forall x, y, t \in S)(xt \neq yt \Rightarrow (x \in xtS \cup ytS \vee y \in xtS \cup ytS)).$$

Proof. **Necessity.** If all right Rees factor acts are principally weakly homoflat then all right Rees factor acts are principally weakly flat and hence we get regularity from Corollary 3.5. Let us show that

$$K_{TF}(xt, yt) = xtS \cup ytS.$$

Inclusion $xtS \cup ytS \subseteq K_{TF}(xt, yt)$ being evident let us show that $K_{TF}(xt, yt) \subseteq xtS \cup ytS$. Take $w \in K_{TF}(xt, yt) = K_{TF}(xt) \cup K_{TF}(yt)$. Without loss of generality we may assume that $w \in K_{TF}(xt)$. By definition of $K_{TF}(xt)$ there exist elements $z, w', r_1, \ldots, r_m, s_1, \ldots, s_{m-1} \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned} s_1 c_1 &= xt r_1 \\ s_2 c_2 &= s_1 r_2 \\ &\cdots \\ z c_m &= s_{m-1} r_m \end{aligned}$$

and $w = zw'$. Since S is regular, every right cancellable element of S is right invertible. Consequently, multiplying the corresponding equalities by the right inverses of right cancellable elements on the right we obtain $s_1 \in xtS$, $s_2 \in xtS, \ldots, z \in xtS$ and so $w \in xtS$. Thus, indeed, $K_{TF}(xt, yt) = xtS \cup ytS$. The rest follows now by Corollary 3.7.

Sufficiency follows from corollaries 3.4 and 3.7. ∎

Example 10 Consider again a semilattice $S = \{1, e, f, 0\}$ with $ef = 0$ from Example 1. This monoid is regular and hence left almost regular. By Corollary 3.5 all right Rees factor acts of S are principally weakly flat and hence all torsion free right Rees factor acts are principally weakly flat.

Since 1 is the only right cancellable element of S, we see that $K_{TF}(e) = eS = \{e, 0\}$ and $K_{TF}(0) = 0S = \{0\}$. Therefore

$$K_{TF}(e, 0) = K_{TF}(e) \cup K_{TF}(0) = \{e, 0\}.$$

Now $1e \neq fe$ and $1, f \notin K_{TF}(1e, fe) = K_{TF}(e, 0) = \{e, 0\}$. This means by Corollary 3.7 that not all torsion free right Rees factor acts are principally weakly homoflat and hence not all right Rees factor acts are principally weakly homoflat.

Now let us consider cyclic acts.

Proposition 3.9 *All cyclic right S-acts are principally weakly homoflat if and only if S is regular and*

$$(\forall x, y, t \in S)(\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y).$$

Proof. **Necessity.** Regularity follows by Corollary 3.5. Suppose that $x, y, t \in S$. By assumption $S/\rho(xt, yt)$ is principally weakly homoflat and hence $xt\rho(xt, yt)yt$ implies by Lemma 2.7 the existence of $u, v \in S$ such that $ut = vt$, $u\rho(xt, yt)x$ and $v\rho(xt, yt)y$.

Sufficiency. Suppose that $xt\rho yt$ for $x, y, t \in S$ and a right congruence ρ on S. By assumption there exist $u, v \in S$ such that $ut = vt$, $u\rho(xt, yt)x$ and $v\rho(xt, yt)y$. Since $\rho(xt, yt) \subseteq \rho$, we have $u\rho x$ and $v\rho y$. Hence S/ρ is principally weakly homoflat by Lemma 2.7. ∎

As we saw in Example 10, the class of monoids described by the condition of Proposition 3.9 is a proper subclass of the class of regular monoids.

For arbitrary right S-acts we can prove the following result.

Proposition 3.10 *All torsion free right S-acts are principally weakly homoflat if and only if S is a right cancellative monoid.*

Proof. **Necessity.** Take an arbitrary element $s \in S$ and the right ideal $K = K_{TF}(s)$. If x, y and z denote elements not belonging to S, define

$$A(K) = (\{x, y\} \times (S \setminus K)) \cup (\{z\} \times K),$$

and define a right S-action on $A(K)$ by

$$\begin{aligned}
(x,u)s &= \begin{cases} (x, us) & \text{if } us \notin K \\ (z, us) & \text{if } us \in K \end{cases} \\
(y,u)s &= \begin{cases} (y, us) & \text{if } us \notin K \\ (z, us) & \text{if } us \in K \end{cases} \\
(z,u)s &= (z, us),
\end{aligned}$$

We obtain a right S-act. (Note that if $K = S$ then $A(K)$ is isomorphic to S_S.) Our first aim is to show that this S-act is torsion free. To this end, suppose that $(a, u)c = (a', v)c$, $a, a' \in \{x, y, z\}$, $u, v, c \in S$, c is right cancellable. Then $uc = vc$ and cancelling c yields $u = v$. If $a = a' = x$ or $a = a' = y$ then $(a, u) = (a', v)$. Otherwise $(a, u)c = (a', v)c = (z, uc)$ and $uc \in K$. That means there exist $t \in L(s)$, $z \in S$ such that $uc = tz$. Since $t \in L(s)$, there exist elements $r, r_1, \ldots, r_m, s_1, \ldots, s_{m-1} \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$ such that

$$\begin{aligned}
s_1 c_1 &= s r_1 \\
s_2 c_2 &= s_1 r_2 \\
&\cdots \\
t c_m &= s_{m-1} r_m.
\end{aligned}$$

These equalities together with $uc = tz$ mean that $u = v \in K$. Consequently $a = a' = z$ and hence again $(a, u) = (a', v)$. Thus $A(K)$ is indeed torsion free.

By assumption $A(K)$ must be principally weakly homoflat. Now the equality $(x, 1)s = (y, 1)s$ $(= (z, s)$, because $s \in K)$ implies the existence of $u, v, w \in S$ and $a \in \{x, y, z\}$ such that $us = vs$, $(x, 1) = (a, w)u$ and $(y, 1) = (a, w)v$ by Proposition 2.6. This implies $x = a = y$, which means $K = S$. Since $1 \in K$, $1 = tz$ and

$$\begin{aligned}
s_1 c_1 &= s r_1 \\
s_2 c_2 &= s_1 r_2 \\
&\cdots \\
s_{m-1} c_{m-1} &= s_{m-2} r_{m-1} \\
t c_m &= s_{m-1} r_m
\end{aligned}$$

for some $t, z, r_1, \ldots, r_m$, $s_1, \ldots, s_{m-1} \in S$ and right cancellable elements $c_1, \ldots, c_m \in S$. Since t and c_m are right cancellable, s_{m-1} is right cancellable. Since s_{m-1} and c_{m-1} are right cancellable, s_{m-2} is right cancellable. Continuing in this manner we get that s is right cancellable.

Sufficiency. Let S be a right cancellative monoid, A_S a torsion free S-act and $as = a's$, $a, a' \in A_S$, $s \in S$. Then $a = a'$ by torsion freeness and we see that A_S is weakly homoflat by taking $a'' = a$, $u = v = 1$ in Proposition 2.6. ∎

Corollary 3.11 *The following assertions are equivalent for a monoid S:*

1. *All torsion free right S-acts are weakly pullback flat.*

2. *All torsion free right S-acts satisfy condition* (P).

3. *All torsion free right S-acts are weakly homoflat.*

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. is obvious.

3. $\Rightarrow$ 1. Assume that all torsion free right S-acts are weakly homoflat. Then S is right cancellative by Proposition 3.10. Suppose that A_S is torsion free and $as = a's'$, $a, a' \in A_S$, $s, s' \in S$. By assumption A_S is weakly homoflat and hence by Lemma 2.11 (using right cancellativity) there exist $a'', a_1, a_2 \in A_S$, $u, v, p_1, p_2, q_1, q_2 \in S$ such that either

$$us = vs',\ a = a''u \text{ and } a' = a''v,$$

or

$$a' = a_2 p_2,\ a = a_2 q_2 \text{ and } q_2 s = p_2 s',$$

or

$$a = a_2 p_2,\ a' = a_2 q_2 \text{ and } p_2 s = q_2 s'.$$

Thus A_S satisfies condition (P). Suppose $as = as'$ and $sz = s'z$, $a \in A_S$, $s, s', z \in S$. Then $s = s'$ and hence $a = a1$ and $1s = 1s'$. Thus A_S satisfies condition (E$'$). By Proposition 2.17 A_S is weakly pullback flat. ∎

Unfortunately we do not have the internal description of monoids for which the assertions of the previous corollary hold. However, we can show that the corresponding class (which is a subclass of the class of right cancellative monoids by Proposition 3.10) is strictly bigger than the class of groups.

Example 11 Let S be a right cancellative monoid such that for all $s, s' \in S$ either $s \in Ss'$ or $s' \in Ss$ (for instance a free monogenic monoid). Let us show that all torsion free right S-acts satisfy condition (P). Suppose that A_S is a torsion free right S-act and $as = a's'$, $a, a' \in A_S$, $s, s' \in S$. By assumption there exists $v \in S$ such that (without loss of generality) $s = vs'$. The equality $avs' = a's'$ implies $av = a'$ by torsion freeness of A_S. Denoting $a'' = a$ we have $a = a''1$, $a' = a''v$ and $1s = vs'$, thus A_S satisfies condition (P).

In [30] it was proved that all right S-acts satisfy condition (P) if and only if S is a group. The proposition below is a little stronger.

Proposition 3.12 *The following assertions are equivalent for a monoid S.*

1. *All right S-acts are weakly pullback flat.*
2. *All right S-acts satisfy condition* (P).
3. *All right S-acts are weakly homoflat.*
4. *All right S-acts are principally weakly homoflat.*
5. *S is a group.*

Proof. 1. $\Rightarrow$ 2. $\Rightarrow$ 3. $\Rightarrow$ 4. is clear.

4. $\Rightarrow$ 5. If all right S-acts are principally weakly homoflat then all right S-acts are principally weakly flat and all torsion free right S-acts are principally weakly homoflat. Hence by Corollary 3.5 S is regular and by Proposition 3.10 S is right cancellative. So S must be a group.

5. $\Rightarrow$ 1. Let S be a group and A_S a right S-act. We know that A_S satisfies condition (P). By Proposition 2.17 it is sufficient to show that A_S satisfies condition (E$'$). Suppose that $as = as'$ and $sz = s'z$ for some $a \in A_S, s, s', z \in S$. Then multiplying $sz = s'z$ by z^{-1} on the right, we get $s = s'$ and hence we can take $a' = a$ and $u = 1$. ∎

Finally we consider the special case of idempotent monoids. For them the condition of Corollary 3.8 takes a simpler form.

Proposition 3.13 *Let S be an idempotent monoid. Then all right Rees factor acts of S are principally weakly homoflat if and only if*

$$(\forall e, f \in S)(ef = f \vee e = efe).$$

Proof. **Necessity.** Take $e, f \in S$. If $e = 1$ or $f = 1$ then we obviously have what we need. Assume that $e \neq 1$, $f \neq 1$ and $ef \neq f$. By Corollary 3.8 either $e \in efS \cup fS$ or $1 \in efS \cup fS$. Since ef and f are not right invertible, necessarily $e \in efS \cup fS$. Suppose $e = efz$ for some $z \in S$. Then

$$e = efz = (ef)(ef)z = (ef)(efz) = efe.$$

If $e = fz$ for some $z \in S$ then

$$e = ee = (fz)(fz) = (fz)f(fz) = efe.$$

Sufficiency. We use Corollary 3.8 to show that all Rees factors of S are principally weakly homoflat. Suppose that $xt \neq yt$, $x, y, t \in S$. It is impossible that $xt = t$ and $yt = t$. Hence either $x = xtx$ or $y = yty$ and thus either $x \in xtS \cup ytS$ or $y \in xtS \cup ytS$. ∎

3.3 Weak homoflatness

Here we characterize monoids over which all (all torsion free, all principally weakly flat, all principally weakly homoflat) right Rees factor acts are weakly homoflat. As a special case we find a description of the idempotent monoids over which all (all principally weakly homoflat) right Rees factor acts are weakly homoflat, and over which all cyclic right S-acts are weakly homoflat.

We start again with Rees factor acts.

Proposition 3.14 *All principally weakly homoflat right Rees factor acts of S are weakly homoflat if and only if S is right reversible and every left stabilizing and left annihilating right ideal of S is strongly left annihilating.*

Proof. **Necessity.** If all principally weakly homoflat right Rees factor acts are weakly homoflat then the one-element right S-act Θ_S is also weakly homoflat and hence weakly flat. By Corollary 1.12 S is right reversible. Suppose that K is a left stabilizing and left annihilating right ideal of S. Then S/K is principally weakly homoflat by Lemma 2.8. By assumption S/K is weakly homoflat. Hence K is strongly left annihilating by Lemma 2.13.

Sufficiency. Suppose S/K is principally weakly homoflat. Then K is left stabilizing and left annihilating by Lemma 2.8. Using assumption and Lemma 2.13 we see that S/K is weakly homoflat. ∎

Corollary 3.15 *If S is a commutative monoid then all principally weakly homoflat right Rees factor acts of S are weakly homoflat.*

Proof. Let K be a left stabilizing and left annihilating right ideal of S and $f : {}_S(Ss \cup St) \to {}_SS$ a homomorphism of left S-acts with $s, t \in S \setminus K$ and $f(s), f(t) \in K$ (note that $s, t \notin K$ implies $1 \notin K$). Then using left annihilation and commutativity we obtain

$$f(s) = tf(s) = f(ts) = f(st) = sf(t) = f(t).$$

Thus K is strongly left annihilating and S/K is weakly homoflat by Lemma 2.13, because a commutative monoid is right reversible. ∎

Example 12 Consider a semilattice $S = \{1, e, f, 0\}$ with $ef = 0$ (as in examples 1 and 10). Since S is commutative, all principally weakly homoflat right Rees factor acts of S are weakly homoflat by Corollary 3.15. But not all principally weakly flat right Rees factor acts are weakly homoflat by Proposition 3.6, because $eS = \{e, 0\}$ is a left stabilizing right ideal which is not left annihilating.

Corollary 3.16 *The following assertions are equivalent for a monoid S:*

1. *All principally weakly flat right Rees factor acts of S are weakly homoflat.*

2. *S is right reversible and every left stabilizing right ideal of S is strongly left annihilating.*

3. *S is right reversible and for all $x, y \in S$, for all homomorphisms $f : {}_S(Sx \cup Sy) \to {}_SS$ such that $x_0 = f(x) \neq f(y) = y_0$, and for all $x_1, y_1, x_2, y_2, \ldots \in S$ such that*

$$(\forall i \in \mathbb{N}_0)(x_{i+1}x_i = x_i) \text{ and } (\forall i \in \mathbb{N}_0)(y_{i+1}y_i = y_i),$$

there exist $p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\}$ and $z \in S$ such that either $x = pz$ or $y = pz$.

Proof. Equivalence of 1 and 2 follows from Propositions 3.6 and 3.14.

2. $\Rightarrow$ 3. Suppose that

$$x_0 = f(x) \wedge (\forall i \in \mathbb{N}_0)(x_{i+1}x_i = x_i) \wedge y_0 = f(y) \wedge (\forall i \in \mathbb{N}_0)(y_{i+1}y_i = y_i)$$

for some $x, y \in S$, a homomorphism $f : {}_S(Sx \cup Sy) \to {}_SS$ such that $f(x) \neq f(y)$, and elements $x_1, y_1, x_2, y_2, \ldots \in S$. Consider a right ideal

$$K = \left(\bigcup_{i \in \mathbb{N}_0} x_iS\right) \bigcup \left(\bigcup_{i \in \mathbb{N}_0} y_iS\right).$$

For every $k \in K$ there exists $l \in K$ such that $lk = k$, that is K is left stabilizing. By assumption K is strongly left annihilating. Since $f(x), f(y) \in K$ and $f(x) \neq f(y)$, either $x \in K$ or $y \in K$. Thus either $x = pz$ or $y = pz$ for some $p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\}$ and $z \in S$.

3. $\Rightarrow$ 1. Let S/K be principally weakly flat. Then K is left stabilizing by Proposition 1.11. We have to show that K is strongly left annihilating. Suppose that $x_0 = f(x) \in K, y_0 = f(y) \in K$ for some $x, y \in S \setminus K$ and a homomorphism $f : {}_S(Sx \cup Sy) \to {}_SS$. Suppose that $x_0 \neq y_0$. Since K is left stabilizing, there exist $x_1, y_1, x_2, y_2, \ldots \in K$ such that $x_{i+1}x_i = x_i$ and $y_{i+1}y_i = y_i$ for every nonnegative integer i. By assumption there exist $p \in \{x_0, x_1, \ldots\} \cup \{y_0, y_1, \ldots\}$ and $z \in S$ such that either $x = pz$ or $y = pz$. Hence either $x \in K$ or $y \in K$, a contradiction. So we must have the equality $f(x) = f(y)$, that means K is strongly left annihilating. ∎

Example 13 Consider again the monoid $S = \{1, s, t, x, y, 0\}$ from Example 2. We saw that $K = \{0, x, y\}$ is a left stabilizing right ideal which is not strongly left annihilating. Hence not all principally weakly flat right Rees factor acts are weakly homoflat by Corollary 3.16.

Let us show that still all principally weakly flat right Rees factor acts are principally weakly homoflat. Right ideals of S are S, $\{0\}$, $sS = tS = \{s, t, x, y, 0\}$, $xS = \{x, 0\}$, $ys = \{y, 0\}$ and $\{x, y, 0\}$. Clearly, the first three right ideals are left annihilating and it was shown in Example 2 that $\{x, y, 0\}$ is left annihilating. Suppose that $uz, vz \in \{x, 0\}$, $z \in S$, $u, v \in S \setminus \{x, 0\}$. Then either $uz = vz = x$, or $z = 0$. In the second case we obtain $uz = vz = 0$. Hence $\{x, 0\}$, and similarly $\{y, 0\}$, is left annihilating, too. Thus all right ideals of S are left annihilating and so all principally weakly flat right Rees factor acts of S are principally weakly homoflat by Proposition 3.6.

The proofs of the following two corollaries are similar to the proofs of Corollaries 3.7 and 3.8.

Corollary 3.17 *All torsion free right Rees factor acts of a monoid S are weakly homoflat if and only if S is left almost regular and for all $x, y \in S$ and all homomorphisms $f : {}_S(Sx \cup Sy) \to {}_SS$*

$$f(x) \neq f(y) \Rightarrow (x \in K_{TF}(f(x), f(y)) \vee y \in K_{TF}(f(x), f(y))).$$

Corollary 3.18 *All right Rees factor acts of a monoid S are weakly homoflat if and only if S is regular, right reversible and for all $x, y \in S$ and all homomorphisms $f : {}_S(Sx \cup Sy) \to {}_SS$*

$$f(x) \neq f(y) \Rightarrow (x \in f(x)S \cup f(y)S \vee y \in f(x)S \cup f(y)S).$$

Our next aim is to find out when all cyclic right acts are weakly homoflat.

Proposition 3.19 *All cyclic right S-acts are weakly homoflat if and only if S is regular and*

$$(\forall x, y, t \in S)(\exists u, v \in S)(ut = vt \wedge u\rho(xt, yt)x \wedge v\rho(xt, yt)y)$$

and for all $s, t \in S$ and all homomorphisms $f : {}_S(Ss \cup St) \to {}_SS$ there exist $u, v, z, w \in S$ such that

$$\begin{aligned}
&(uf(s) = vf(t) \wedge us\tau s \wedge vt\tau t) \vee \\
&(uf(s) = vf(s) \wedge u\tau 1 \wedge z\tau v \wedge zs = wt\tau t) \vee \\
&(uf(t) = vf(t) \wedge v\tau 1 \wedge w\tau u \wedge zs = wt\tau s)
\end{aligned}$$

where $\tau = \rho(f(s), f(t))$.

Proof. **Necessity.** Proposition 3.9 implies that S is regular and the first condition holds. Take $s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to S$. By assumption $S/\rho(f(s), f(t))$ is weakly homoflat and hence by Lemma 2.12 there exist $u, v, p_1, p_2, q_1, q_2 \in S$ such that either $f(us) = f(vt)$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [u] & p_1 s &= q_1 s \\ [1] &= [p_2] & & \\ [q_2] &= [v] & p_2 t &= q_2 t, \end{aligned}$$

or $f(us) = f(vs)$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 t &= q_1 t \\ [q_2] &= [v] & p_2 t &= q_2 s \\ [1] &= [u], & & \end{aligned}$$

or $f(ut) = f(vt)$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 s &= q_1 s \\ [q_2] &= [u] & p_2 s &= q_2 t \\ [1] &= [v]. & & \end{aligned}$$

In the first case $us\rho(f(s), f(t))s$ and $vt\rho(f(s), f(t))t$, in the second case $q_2 s = p_2 t\rho(f(s), f(t))t$ and in the third case $p_2 s = q_2 t\rho(f(s), f(t))s$.

Sufficiency. Let ρ be a right congruence on a monoid S and let $f(s)\rho f(t)$ for $s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_S S$. By assumption there exist $u, v, z, w \in S$ such that

$$\begin{aligned} &(uf(s) = vf(t) \wedge us\tau s \wedge vt\tau t)\vee \\ &(uf(s) = vf(s) \wedge u\tau 1 \wedge z\tau v \wedge zs = wt\tau t)\vee \\ &(uf(t) = vf(t) \wedge v\tau 1 \wedge w\tau u \wedge zs = wt\tau s) \end{aligned}$$

where $\tau = \rho(f(s), f(t)) \subseteq \rho$.

In the first case for $u, 1, s$ there exist $q_1, p_1 \in S$ such that $q_1 s = p_1 s$, $q_1\rho(us, s)u$ and $p_1\rho(us, s)1$. Now $us\tau s$ implies $\rho(us, s) \subseteq \tau \subseteq \rho$, thus $q_1\rho u$ and $p_1\rho 1$. Analogously using $vt\tau t$ we get $q_2, p_2 \in S$ such that $q_2\rho v$, $p_2\rho 1$ and $p_2 t = q_2 t$.

In the second case using $wt\tau t$ we get $q_1, p_1 \in S$ such that $q_1\rho w$, $p_1\rho 1$ and $p_1 t = q_1 t$.

In the third case $zs\tau s$ implies the existence of $q_1, p_1 \in S$ such that $q_1\rho z$, $p_1\rho 1$ and $p_1 s = q_1 s$.

Thus S/ρ is weakly homoflat by Lemma 2.12. ∎

Finally, we consider the case of idempotent monoids.

Proposition 3.20 *Let S be an idempotent monoid. All principally weakly homoflat right Rees factor acts of S are weakly homoflat if and only if S is left collapsible and every left annihilating right ideal is strongly left annihilating.*

Proof. **Necessity.** By Proposition 3.14 S is right reversible. It is easy to see that an idempotent monoid is right reversible if and only if it is left collapsible. Suppose that K is a left annihilating right ideal. Since S is an idempotent monoid, K is left stabilizing. Again by Proposition 3.14 K is strongly left annihilating.

Sufficiency. Let S/K be principally weakly homoflat. Then K is left annihilating by Lemma 2.8. By assumption K is strongly left annihilating. Hence S/K is weakly homoflat by Lemma 2.13. ∎

Corollary 3.21 *Let S be an idempotent monoid. All right Rees factor acts of S are weakly homoflat if and only if S is left collapsible and*

$$(\forall x, y \in S)(\forall s, t \in S \setminus (xS \cup yS))$$
$$(x \neq y \wedge sx = x \wedge ty = y \Rightarrow (\exists u, v \in S)(us = vt \wedge ux \neq vy)).$$

Proof. **Necessity.** Left collapsibility follows from Proposition 3.20. Suppose that

$$(\exists x, y \in S)(\exists s, t \in S \setminus (xS \cup yS))$$
$$(x \neq y \wedge sx = x \wedge ty = y \wedge (\forall u, v \in S)(us \neq vt \vee ux = vy)).$$

Denote $K = xS \cup yS$. By assumption S/K is weakly homoflat and hence K is strongly left annihilating. Define a mapping $f : {}_S(Ss \cup St) \to {}_SS$ by

$$\begin{aligned} f(us) &= ux, \\ f(ut) &= uy, \end{aligned}$$

$u \in S$. If $us = vs$, $u, v \in S$ then $sx = x$ implies $ux = vx$. Analogously $ut = vt$, $u, v \in S$ implies $uy = vy$. If $us = vt$, $u, v \in S$ then $ux = vy$. Thus f is well defined and clearly it is a homomorphism of left S-acts. Now $s, t \in S \setminus K$, $f(s), f(t) \in K$, but $f(s) \neq f(t)$. So K is not strongly left annihilating, a contradiction.

Sufficiency. Let K be a right ideal of S. Since S is idempotent, K is left stabilizing. Suppose that $f(s), f(t) \in K$ for some $s, t \in S \setminus K$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. Denote $x = f(s)$ and $y = f(t)$. Then $s, t \in S \setminus (xS \cup yS)$, $sx = x$ and $ty = y$. Suppose $x \neq y$. Then by assumption there exist $u, v \in S$ such that $us = vt$, but $ux \neq vy$, that is, $f(us) \neq f(vt)$, a contradiction. Consequently $x = y$, or $f(s) = f(t)$. This means that K is strongly left annihilating. By Lemma 2.13 S/K is weakly homoflat. ∎

Recall that the monoid S in Example 12 was an idempotent monoid. Thus, Corollary 3.21 describes a strictly smaller class of monoids than Proposition 3.20.

It also can be seen that Corollary 3.21 describes a strictly smaller class of monoids than Proposition 3.13. Indeed, if T is a right zero semigroup with two or more elements and $S = T^1$, then for all $e, f \in S$ either $ef = f$ (in the case $f \neq 1$) or $e = efe$ (in the case $f = 1$). On the other hand, the condition of Corollary 3.21 is not satisfied, because S is not a left collapsible monoid.

Now let us consider cyclic acts of idempotent monoids. First let us recall some facts about idempotent monoids.

An idempotent semigroup is also called a *band*. A band S is called *rectangular*, if $efe = e$ for all $e, f \in S$. Every rectangular band is isomorphic to a cartesian product $I \times \Lambda$, with multiplication given by

$$(i, \lambda)(j, \mu) = (i, \mu),$$

$i, j \in I$, $\lambda, \mu \in \Lambda$. Every band S is a semilattice of rectangular bands (see, e.g. [10]), that is

$$S = \bigcup_{\gamma \in \Gamma} S_\gamma$$

where each $S_\gamma \subseteq S$ is a rectangular band, $\gamma \neq \delta$ implies $S_\gamma \cap S_\delta = \emptyset$, Γ is a lower semilattice and

$$S_\gamma S_\delta \subseteq S_{\gamma\delta}$$

for all $\gamma, \delta \in \Gamma$ (we use the notation of multiplication for the operation of this semilattice). Since an idempotent monoid is a band, it is also a semilattice of rectangular bands.

For a special case of rectangular bands with identity adjoined we have the following result.

Lemma 3.22 *Let S be a rectangular band with identity adjoined. If all principally weakly homoflat cyclic right S-acts are weakly homoflat then S is a left zero semigroup with identity adjoined.*

Proof. Let $S = (I \times \Lambda)^1$ and let all principally weakly homoflat cyclic right S-acts be weakly homoflat. Suppose that $|\Lambda| \geq 2$. Choose $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$ and an arbitrary element $i \in I$. Denote $s = (i, \lambda)$ and $t = (i, \mu)$. Let us show that

$$\rho(s, t) = \{(s, t), (t, s)\} \cup \{(z, z) \mid z \in S\}.$$

The converse being obvious let us show that $\rho(s,t)$ is contained in the set which is on the right-hand side of the last equality. For this, suppose that $(x,y) \in \rho(s,t)$, $x,y \in S$. By Lemma 1.15 either $x = y$ or there exist a natural number n and elements $y_1,\ldots,y_n$, $s_1,\ldots,s_n$, $t_1,\ldots,t_n \in S$ such that

$$\begin{aligned} x &= s_1y_1 & t_2y_2 &= s_3y_3 & & \\ t_1y_1 &= s_2y_2 & & & \ldots\ t_ny_n &= y, \end{aligned}$$

where $\{s_i,t_i\} = \{s,t\}$ for every $i \in \{1,\ldots,n\}$. Let this sequence of equalities be the shortest. If none of the elements $y_1,\ldots,y_n$ is equal to 1 then using the multiplication rule in $I \times \Lambda$ we obtain

$$x = s_1y_1 = t_1y_1 = \ldots = t_ny_n = y.$$

Otherwise let j be the smallest index such that $y_j = 1$. If $j \geq 2$ then $x = t_{j-1}y_{j-1} = s_jy_j = s_j$ and the sequence can be shortened. If $j = 1$ then $x = s_1y_1 = s_1$ and $t_1 = t_1y_1 = s_2y_2$. If $y_2 = 1$ then $t_1 = s_2$ and $s_1 = t_2$, hence $x = s_1 = t_2 = t_2y_2$. If $y_2 \neq 1$ then $t_1 = s_2y_2 = t_2y_2 = s_3y_3$. So whenever $n \geq 2$, the sequence can be shortened. Consequently $n = 1$, that is $x = s_1$ and $y = t_1$. Therefore either $(x,y) = (s,t)$ or $(x,y) = (t,s)$.

Now let us show that $S/\rho(s,t)$ is principally weakly homoflat. Suppose that $xp\rho(s,t)yp$ for some $x,y,p \in S$. If $p = 1$ then $xp = xp$, $x\rho(s,t)x$ and $x\rho(s,t)y$, hence S/ρ is principally weakly homoflat by Lemma 2.7. If $p \neq 1$ then $(xp,yp) \neq (s,t)$ and $(xp,yp) \neq (t,s)$, hence $xp = yp$ and, again, S/ρ is principally weakly homoflat by Lemma 2.7. By assumption S/ρ is weakly homoflat and hence there exist $u,v \in S$ such that $us = vt$, but this contradicts the choice of s and t. Thus $|\Lambda| = 1$ which means that $I \times \Lambda$ is a left zero band. ∎

It will turn out soon (see Corollary 3.24) that all cyclic right S-acts over a left zero band with identity adjoined are weakly homoflat.

If $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ is a chain Γ of semigroups S_γ then it is called a *left annihilating chain* if

$$(\forall\gamma,\delta \in \Gamma)(\forall s \in S_\gamma)(\forall t \in S_\delta)(\gamma > \delta \Rightarrow st = t).$$

Similarly right annihilating chains of semigroups are defined. An *annihilating chain* is left and right annihilating chain.

A band S is called *left regular* if $efe = ef$ for all $e,f \in S$. For a characterization of idempotent monoids over which all cyclic right acts are weakly homoflat we need the following theorem.

Theorem 3.23 ([5]) *Let S be an idempotent monoid. All cyclic right S-acts are weakly flat if and only if S is a left regular band.*

Corollary 3.24 *The following assertions are equivalent for an idempotent monoid S:*

1. *All cyclic right S-acts are weakly homoflat.*
2. $$(\forall e, f \in S)((ef = f \vee e = efe) \wedge efe = ef).$$
3. *S is an annihilating chain of left zero bands.*

Proof. 1. $\Rightarrow$ 2. This follows immediately from Proposition 3.13 and Theorem 3.23, because weak homoflatness implies weak flatness.

2. $\Rightarrow$ 3. Let $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ be a representation of S as a semilattice Γ of rectangular bands S_γ. Take any $\gamma, \delta \in \Gamma$ and $e \in S_\gamma, f \in S_\delta$. By assumption either $ef = f$ or $e = efe$. In the first case $\gamma \geq \delta$ and in the second case $\gamma \leq \delta$. Thus Γ is a chain. Let $e \in S_\gamma, f \in S_\delta$, $\gamma, \delta \in \Gamma$, $\gamma > \delta$. Then $e = efe$ is impossible, thus $ef = f$ and S is a left annihilating chain. Using left regularity we get $f = ef = efe$. Therefore $f = fe$, that is S is a right annihilating chain, too. Hence S is an annihilating chain of rectangular bands. Further, take $e, f \in S_\gamma$, $\gamma \in \Gamma$. If $ef = f$, then using rectangularity of S_γ and left regularity of S we obtain

$$e = efe = ef = f.$$

Otherwise,

$$e = efe = ef.$$

Thus, S_γ, $\gamma \in \Gamma$ is a left zero band and S is an annihilating chain of left zero bands.

3. $\Rightarrow$ 1. Let $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ be an annihilating chain Γ of left zero bands S_γ and ρ a right congruence on S. We use Lemma 2.12 to check that S/ρ is weakly homoflat. Suppose that $f(s)\rho f(t)$ for some $s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. Let $s \in S_{\alpha_1}$, $t \in S_{\alpha_2}$, $f(s) \in S_{\beta_1}$, $f(t) \in S_{\beta_2}$. Then $\alpha_1 \geq \beta_1$, because $sf(s) = f(s)$, and, analogously, $\alpha_2 \geq \beta_2$. Since Γ is a chain, without loss of generality we may assume that $\alpha_1 \geq \alpha_2$. This means that $t = ts$. If $\alpha_1 = \beta_1$ then

$$f(s) = sf(s) = s$$

and, since $\beta_1 = \alpha_1 \geq \alpha_2$,

$$f(t) = f(ts) = tf(s) = t.$$

Thus $tf(s) = tf(t)$ and

$$\begin{array}{rcll} [1] & = & [1] & \\ [s] & = & [t] & 1s = ss \\ [1] & = & [1] & \\ [t] & = & [t] & 1t = tt. \end{array}$$

Now suppose that $\alpha_1 > \beta_1$ and consider the following cases.

a) $\beta_1 \geq \alpha_2$. Then, as before, $f(t) = f(ts) = tf(s) = t$. Consequently $1f(s) = f(s)f(s)$ and

$$\begin{array}{rcll} [1] & = & [1] & \\ [t] & = & [t] & 1t = tt \\ [t] & = & [f(s)] & tt = ts \\ [1] & = & [1]. & \end{array}$$

b) $\alpha_2 > \beta_1$. Then $f(s) = tf(s) = f(ts) = f(t)$. Hence $1f(s) = 1f(t)$ and

$$\begin{array}{rcll} [1] & = & [1] & \\ [1] & = & [1] & 1s = 1s \\ [1] & = & [1] & \\ [1] & = & [1] & 1t = 1t. \end{array}$$

This means that S/ρ is weakly homoflat. ∎

For arbitrary right acts over an idempotent monoid we have the following result.

Proposition 3.25 *The following assertions are equivalent for an idempotent monoid S:*

1. *All right S-acts are weakly homoflat.*
2. *All right S-acts are principally weakly homoflat.*
3. *$S = \{1\}$.*

Proof. 1. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 3. By Corollary 3.12 S is a group. The only idempotent group is the trivial group $S = \{1\}$.

3. $\Rightarrow$ 1. If $S = \{1\}$ then all right S-acts are even free. ∎

3.4 Condition (P)

Here we give a characterization of monoids over which all right Rees factor acts with some weaker property than condition (P) have condition (P).

Proposition 3.26 *All weakly homoflat right Rees factor acts of S satisfy condition* (P) *if and only if S is not right reversible or no nontrivial right ideal of S is left stabilizing and strongly left annihilating.*

Proof. Necessity. Suppose that S is right reversible and K is a nontrivial left stabilizing and strongly left annihilating right ideal. Then S/K is weakly homoflat by Lemma 2.13. By assumption S/K must satisfy condition (P), but this means that K is trivial by Proposition 1.11, a contradiction.

Sufficiency. Let S/K be weakly homoflat. Then S is right reversible by Lemma 2.13. If $K = S$, that is $S/K \cong \Theta_S$, then S/K satisfies condition (P) by Lemma 2.14. If $|K| = 1$ then $S/K \simeq S$ is free and hence satisfies condition (P). Suppose that K is a nontrivial right ideal. By assumption K is either not left stabilizing or not strongly left annihilating. Thus S/K is not weakly homoflat by Lemma 2.13, a contradiction. ∎

Corollary 3.27 *All principally weakly homoflat right Rees factor acts of S satisfy condition* (P) *if and only if S is right reversible and no nontrivial right ideal of S is left stabilizing and left annihilating.*

Proof. Necessity. By Proposition 3.14 S is right reversible. Suppose K is a nontrivial left stabilizing and left annihilating ideal of S. Then S/K is principally weakly homoflat by Lemma 2.8. By assumption S/K satisfies condition (P), so K must be trivial, a contradiction.

Sufficiency. Let S/K be principally weakly homoflat. Then K is left stabilizing and left annihilating. By assumption K must be trivial. Hence S/K satisfies condition (P) by Proposition 1.11. ∎

The following is a corollary of Corollary 3.27 and Proposition 3.6.

Corollary 3.28 ([20]) *All principally weakly flat right Rees factor acts of S satisfy condition* (P) *if and only if S is right reversible and no nontrivial right ideal is left stabilizing.*

Corollary 3.29 *All torsion free right S-acts satisfy condition* (P) *if and only if S is right cancellative monoid with a zero adjoined, or right cancellative and right reversible.*

Proof. **Necessity.** By Corollary 3.28 S is right reversible. Take $s \in S$. Then the right Rees factor act $S/K_{TF}(s)$ is torsion free. By assumption $S/K_{TF}(s)$ satisfies condition (P). By Proposition 1.11 $|K_{TF}(s)| = 1$ or $K_{TF}(s) = S$ is right reversible. In the first case s is a left zero, because $sS \subseteq K_{TF}(s)$. In the second case $1 \in K_{TF}(s)$ implies that s is right cancellable as in the proof of Proposition 3.10. So every element of S is either a left zero or right cancellable. Let K be the set of all left zeros of S. If K is nonempty then it is a left stabilizing right ideal and by Corollary 3.28 it must be trivial. Thus S has at most one left zero which must then be a zero. Since the product of two right cancellable elements is right cancellable, we have $S = C^0$ in the case if S has a zero 0.

Sufficiency. Let S be a right cancellative monoid with a zero adjoined, or right cancellative and right reversible monoid. Suppose that S/K is torsion free for a right ideal K of S. If $|K| = 1$ then S/K is even free. Otherwise K contains a right cancellable element c. Hence $1c \in K$ implies by torsion freeness of S/K that $1 \in K$, that is $K = S$. Thus S/K satisfies again condition (P). ∎

Since a regular right cancellable element of a monoid must be right invertible, Corollaries 3.18 and 3.29 imply the following result.

Corollary 3.30 ([20]) *All right Rees factor acts of S satisfy condition* (P) *if and only if S is a group or a group with a zero adjoined.*

Now let us consider cyclic acts.

For given $s, t \in S$ let us define a sequence of subsets of $S \times S$

$$\begin{aligned} F_1 &= \{(x,y) \mid (\exists c \in S)(c \text{ is right cancellable and } (xc, yc) \in \rho(s,t))\}, \\ F_{i+1} &= \{(x,y) \mid (\exists c \in S)(c \text{ is right cancellable and } (xc, yc) \in \rho(F_i))\} \end{aligned}$$

and a binary relation on S

$$\rho_{TF}(s,t) = \bigcup_{i=1}^{\infty} \rho(F_i).$$

Clearly, ρ is a right congruence on S.

Lemma 3.31 *All cyclic torsion free right S-acts satisfy condition* (P) *if and only if*

$$(\forall s, t \in S)(\exists u, v \in S)(us = vt \wedge u \rho_{TF}(s,t) 1 \wedge v \rho_{TF}(s,t) 1).$$

Proof. **Necessity** Suppose $(xc, yc) \in \rho_{TF}(s,t)$ for $x, y, c \in S$ where c is right cancellable. Then there exists a natural number k such that $(xc, yc) \in \rho(F_k)$. By the definition of F_{k+1} we have $(x, y) \in F_{k+1}$ and hence $(x, y) \in \rho(F_{k+1}) \subseteq \rho_{TF}(s,t)$. This means that $S/\rho_{TF}(s,t)$ is torsion free. The rest follows now from Proposition 1.10, because $S/\rho_{TF}(s,t)$ has to satisfy condition (P) and $(s,t) \in \rho_{TF}(s,t)$.

Sufficiency. Let ρ be a right congruence on S and $s\rho t$, $s, t \in S$. Then $\rho(s,t) \subseteq \rho$. If $(xc, yc) \in \rho(s,t) \subseteq \rho$, $x, y, c \in S$, c is right cancellable, then using torsion freeness of S/ρ we obtain $(x, y) \in \rho$. Hence $F_1 \subseteq \rho$ and thus $\rho(F_1) \subseteq \rho$ because $\rho(F_1)$ is the smallest right congruence containing F_1. Assume that $\rho(F_i) \subseteq \rho$. Suppose that $(xc, yc) \in \rho(F_i) \subseteq \rho$, $x, y, c \in S$, c is right cancellable. Then using torsion freeness of S/ρ we obtain $(x, y) \in \rho$. Hence $F_{i+1} \subseteq \rho$ and thus $\rho(F_{i+1}) \subseteq \rho$. So we have shown that $\rho_{TF}(s,t) \subseteq \rho$. By assumption there exist $u, v \in S$ such that $us = vt$, $u\rho_{TF}(s,t)1$ and $v\rho_{TF}(s,t)1$. But then $us = vt$, $u\rho 1$ and $v\rho 1$, which means that S/ρ satisfies condition (P). ∎

3.5 Weak pullback flatness

Here we try to answer the questions "When are all right Rees factor acts with property X weakly pullback flat?".

Proposition 3.32 *All right Rees factor acts of S satisfying condition* (P) *are weakly pullback flat if and only if S is not right reversible or S is weakly left collapsible.*

Proof. **Necessity.** Suppose that S is right reversible. Then the one-element right S-act Θ_S satisfies condition (P). By assumption Θ_S is weakly pullback flat and hence S is weakly left collapsible by Corollary 2.20.

Sufficiency. Let S/K satisfy condition (P). By Proposition 1.11 either $|K| = 1$ or $K = S$ is right reversible. In the first case S/K is free and hence weakly pullback flat. In the second case we know that S is weakly left collapsible by assumption. Hence S/K is weakly pullback flat by Lemma 2.19. ∎

Using Propositions 3.26 and 3.32 we get the following.

Corollary 3.33 *All weakly homoflat right Rees factor acts of S are weakly pullback flat if and only if S is not right reversible or S is weakly left collapsible and no nontrivial right ideal is left stabilizing and strongly left annihilating.*

The next result comes from Corollaries 3.27 and 3.33.

Corollary 3.34 *All principally weakly homoflat right Rees factor acts of S are weakly pullback flat if and only if S is right reversible and weakly left collapsible and no nontrivial right ideal of S is left stabilizing and left annihilating.*

Corollaries 3.28 and 3.34 yield the following corrollary.

Corollary 3.35 *All principally weakly flat right Rees factor acts of S are weakly pullback flat if and only if S is right reversible and weakly left collapsible and no nontrivial right ideal of S is left stabilizing.*

Corollary 3.36 *All torsion free right Rees factor acts of S are weakly pullback flat if and only if S is a right cancellative monoid with a zero adjoined, or S is right cancellative and right reversible.*

Proof. **Necessity** follows from Corollary 3.29.

Sufficiency. By Corollary 3.29 all torsion free right Rees factor acts satisfy condition (P). By Proposition 3.32 it is sufficient to show that S is weakly left collapsible. Suppose that $sz = s'z$ for some $s, s', z \in S$. If S contains zero then $0s = 0s'$ and we are done. Otherwise S is right cancellative and hence $sz = s'z$ implies $s = s'$ and we have $1s = 1s'$. Thus S is weakly left collapsible. ∎

Corollary 3.37 *All right Rees factor acts of S are weakly pullback flat if and only if S is a group or a group with a zero adjoined.*

Proof. **Necessity** follows from Corollary 3.30.

Sufficiency. By Corollary 3.30 all right Rees factor acts satisfy Condition (P). The rest follows from Proposition 3.32 because groups and groups with a zero adjoined are weakly left collapsible. ∎

To get the following corollary for cyclic acts we need a proposition.

Proposition 3.38 ([21]) *All cyclic right S-acts satisfying condition* (P) *are weakly pullback flat if and only if every right reversible submonoid of S is weakly left collapsible.*

Corollary 3.39 *All cyclic right S-acts are weakly pullback flat if and only if S is a group or $S = \{0, 1\}$.*

Proof. **Necessity.** By Corollary 3.37 S is a group or a group with a zero adjoined. Suppose that $S = G^0$ where G is a nontrivial group. Then G is a right reversible submonoid of S which is not weakly left collapsible.

Hence all cyclic right S acts cannot be weakly pullback flat by Proposition 3.38.

Sufficiency. If S is a group then all S-acts are weakly pullback flat by Proposition 3.12. If $S = \{0, 1\}$ then all cyclic acts are pullback flat (and hence weakly pullback flat) by Theorem 3.1 of [23]. ∎

3.6 Pullback flatness

Here we try to answer the questions "When are all right Rees factor acts with property X pullback flat?". We also study cyclic acts over idempotent monoids.

Proposition 3.40 *All weakly pullback flat right Rees factor acts of S are pullback flat if and only if S is not a right reversible weakly left collapsible monoid or S is left collapsibile.*

Proof. **Necessity.** Suppose that S is a right reversible and weakly left collapsible monoid. Then by Corollary 2.20 the one-element right S-act Θ_S is weakly pullback flat. By assumption Θ_S is pullback flat and hence S is left collapsible by Corollary 1.12.

Sufficiency. Let S/K be a weakly pullback flat right Rees factor act. By Lemma 2.19 either $|K| = 1$ or $K = S$ is right reversible and weakly left collapsible. In the first case S/K is free and hence pullback flat. In the second case S is left collapsible by assumption. Hence $S/K \cong \Theta_S$ is pullback flat by Corollary 1.12. ∎

The following six results are direct consequences of Proposition 3.40, Proposition 3.32 and Corollaries 3.33, 3.34, 3.35, 3.36 and 3.37.

Corollary 3.41 ([20]) *All right Rees factor acts of S satisfying condition* (P) *are pullback flat if and only if S is not right reversible or S is left collapsible.*

Corollary 3.42 *All weakly homoflat right Rees factor acts of S are pullback flat if and only if S is not right reversible or S is left collapsible and no nontrivial right ideal of S is left stabilizing and strongly left annihilating.*

Corollary 3.43 *All principally weakly homoflat right Rees factor acts of S are pullback flat if and only if S is left collapsible and no nontrivial right ideal of S is left stabilizing and left annihilating.*

Corollary 3.44 ([20]) *All principally weakly flat right Rees factor acts of S are pullback flat if and only if S is left collapsible and no nontrivial right ideal of S is left stabilizing.*

Corollary 3.45 *All torsion free right Rees factor acts of S are pullback flat if and only if S is a right cancellative monoid with a zero adjoined, or S is right cancellative and left collapsible.*

Corollary 3.46 ([20]) *All right Rees factor acts of S are pullback flat if and only if S is a group with a zero adjoined or $S = \{1\}$.*

Let us consider again a special case of idempotent monoids. Observe that if S is an idempotent monoid then by Proposition 2.13 of [1] a right S-act is pullback flat if and only if it satisfies condition (P).

Lemma 3.47 *Let S be an idempotent monoid. If all weakly homoflat cyclic right S-acts are pullback flat then S is a semilattice of right zero bands.*

Proof. Let $S = \bigcup_{\gamma\in\Gamma} S_\gamma$ be a semilattice Γ of rectangular bands S_γ and let all weakly homoflat cyclic right S-acts be pullback flat. Suppose that $S_\gamma = I \times \Lambda$, $\gamma \in \Gamma$, $1 \notin S_\gamma$ and $|I| \geq 2$. Choose $i, j \in I$, $i \neq j$ and $\lambda \in \Lambda$. Denote $s = (i, \lambda)$, $t = (j, \lambda)$ and $\rho = \rho(s,t)$. Then $st = s$ and $ts = t$.

First, let us show that S/ρ is weakly homoflat. Suppose that $f(s)\rho f(t)$ for $s, t \in S$ and a homomorphism $f : {}_S(Ss \cup St) \to {}_SS$. By Lemma 1.15 either $f(s) = f(t)$ or there exist a natural number n and elements $y_1, \ldots, y_n$, $s_1, \ldots, s_n$, $t_1, \ldots, t_n \in S$ such that

$$\begin{array}{rclrclrcl} f(s) & = & s_1y_1 & t_2y_2 & = & s_3y_3 & & & \\ & & t_1y_1 & = & s_2y_2 & & \ldots\ t_ny_n & = & f(t), \end{array}$$

where $\{s_i, t_i\} = \{s, t\}$ for every $i \in \{1, \ldots, n\}$. Multiplying all these equalities by s on the left and using the equality $st = s$ we obtain

$$\begin{aligned} sf(s) &= ss_1y_1 = sy_1 = st_1y_1 = ss_2y_2 = sy_2 \\ &= st_2y_2 = \ldots = st_ny_n = sf(t). \end{aligned}$$

Moreover,

$$\begin{array}{rclrcl} [1] & = & [1] & & & \\ [s] & = & [s] & 1s & = & ss \\ [1] & = & [1] & & & \\ [t] & = & [s] & 1t & = & tt, \end{array}$$

i.e. S/ρ is weakly homoflat.

By assumption S/ρ is pullback flat. Therefore by Proposition 1.10 $s\rho t$ implies the existence of $u \in S$ such that $u\rho 1$ and $us = ut$. Since s, t being nonidentity idempotents are not right invertible, $[1]_\rho = \{1\}$. Consequently $u = 1$ and $s = t$, a contradiction. Thus $|I| = 1$, that is, S_γ is a right zero band. ∎

Lemma 3.48 *If S is a right zero band with an identity adjoined then all weakly homoflat cyclic right S-acts are pullback flat.*

Proof. Let $S = T^1$ where T is a right zero band and let ρ be a right congruence on S such that S/ρ is weakly homoflat and suppose that $s\rho t$, $s, t \in S$. If $s = 1$ then $ts = tt$ and $t\rho 1$. Analogous argument applies if $t = 1$. If $s, t \in T$ then using weak homoflatness of S/ρ (taking f the inclusion of ${}_S(Ss \cup St)$ into ${}_SS$) we get the elements $u, v, p_1, p_2, q_1, q_2 \in S$ such that either $us = vt$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [u] & p_1 s &= q_1 s \\ [1] &= [p_2] & & \\ [q_2] &= [v] & p_2 t &= q_2 t, \end{aligned}$$

or $us = vs$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 t &= q_1 t \\ [q_2] &= [v] & p_2 t &= q_2 s \\ [1] &= [u], & & \end{aligned}$$

or $ut = vt$ and

$$\begin{aligned} [1] &= [p_1] & & \\ [q_1] &= [p_2] & p_1 s &= q_1 s \\ [q_2] &= [u] & p_2 s &= q_2 t \\ [1] &= [v]. & & \end{aligned}$$

Since T is a right zero band, we have $s = t$ in any case. Thus $1s = 1t$ and $1\rho 1$ yield that S/ρ is pullback flat. ∎

3.7 Projectivity

Here we try to answer the questions "When are all right Rees factor acts with property X projective?".

Proposition 3.49 ([20]) *All pullback flat right Rees factor acts of S are projective if and only if left S is not left collapsible or S has a left zero.*

The following seven results are direct consequences of Proposition 3.49, Proposition 3.40 and corollaries 3.41, 3.42, 3.43, 3.44, 3.45 and 3.46.

Corollary 3.50 *All weakly pullback flat right Rees factor acts of S are projective if and only if S is not a right reversible weakly left collapsible monoid or S has a left zero.*

Corollary 3.51 ([20]) *All right Rees factor acts of S satisfying condition* (P) *are projective if and only if S is not right reversible or S has a left zero.*

Corollary 3.52 *All weakly homoflat right Rees factor acts of S are projective if and only if S is not right reversible or S has a left zero and no nontrivial right ideal of S is left stabilizing and strongly left annihilating.*

Corollary 3.53 *All principally weakly homoflat right Rees factor acts of S are projective if and only if S has a left zero and no nontrivial right ideal of S is left stabilizing and left annihilating.*

Corollary 3.54 ([20]) *All principally weakly flat right Rees factor acts of S are projective if and only if S has a zero and no nontrivial right ideal of S is left stabilizing.*

Note that the existence of a zero follows from the fact that the subset consisting of all left zeros of S is a left stabilizing right ideal.

Corollary 3.55 *All torsion free right Rees factor acts of S are projective if and only if S is a right cancellative monoid with a zero adjoined or $S = \{1\}$.*

Corollary 3.56 ([20]) *All right Rees factor acts of S are projective if and only if S is a group with a zero adjoined or $S = \{1\}$.*

The following results concern idempotent monoids.

Proposition 3.57 *The following assertions are equivalent for an idempotent monoid S:*

1. *All weakly homoflat right Rees factor acts of S are projective.*
2. *All weakly homoflat right Rees factor acts of S are pullback flat.*
3. *S is not left collapsible or $S = \{1\}$ or $S = \{0, 1\}$.*

Proof. 1. $\Rightarrow$ 2. is obvious.

2. $\Rightarrow$ 3. Let S be left collapsible. Denote $K = S \setminus \{1\}$. If $K = \emptyset$ then $S = \{1\}$. Otherwise K is a left stabilizing strongly left annihilating right ideal. Hence the right Rees factor act S/K is weakly homoflat. By assumption S/K is pullback flat. Consequently $|K| = 1$ by Proposition 1.11. Thus $S = \{0, 1\}$.

3. $\Rightarrow$ 1. Let S/K be weakly homoflat. Then S is left collapsible and hence by assumption $S = \{1\}$ or $S = \{0, 1\}$. By Corollary 3.56 S/K is projective. ∎

For the next corollary we need the notion of right perfect monoids. A monoid is called *right perfect* [11] if every right S-act has a projective cover. It was proved in [7] that a monoid is right perfect if and only if all pullback flat right acts over it are projective. It was shown in [18] that a monoid S is right perfect if and only if S satisfies the conditions (A) and (K) below:

(A) every right S-act satisfies the ascending chain condition for cyclic subacts;

(K) if $P \subseteq S$ is a left collapsible submonoid then P contains a left zero.

Recall that if S is an idempotent monoid then every right S-act satisfying condition (P) is pullback flat.

Corollary 3.58 *The following assertions are equivalent for an idempotent monoid S:*

1. *All principally weakly homoflat right S-acts are projective.*
2. *All principally weakly homoflat right S-acts are pullback flat.*
3. *All principally weakly homoflat cyclic right S-acts are projective.*
4. *All principally weakly homoflat cyclic right S-acts are pullback flat.*
5. *All principally weakly homoflat right Rees factor acts of S are projective.*
6. *All principally weakly homoflat right Rees factor acts of S are pullback flat.*
7. $S = \{1\}$ *or* $S = \{0, 1\}$.

Proof. Implications 1. $\Rightarrow$ 3. $\Rightarrow$ 5., 2. $\Rightarrow$ 4. $\Rightarrow$ 6., 1. $\Rightarrow$ 2., 3. $\Rightarrow$ 4. and 5. $\Rightarrow$ 6. are obvious.

6. $\Rightarrow$ 7. By Proposition 3.20 S is left collapsible. Hence by Proposition 3.57 $S = \{1\}$ or $S = \{0, 1\}$.

7. $\Rightarrow$ 1. For $S = \{1\}$ all right acts are projective. Consider $S = \{0, 1\}$. Let A_S be a principally weakly homoflat right S-act. We shall show that A_S satisfies condition (P) (and hence is pullback flat). Suppose $as = a's'$, $a, a' \in A_S$, $s, s' \in S$. If $s = s'$ then we can simply apply principal weak homoflatness. If, e.g., $s = 0$ and $s' = 1$ then $1 \cdot 0 = 0 \cdot 1$, $a = a1$ and $a' = a0$. Hence A_S satisfies condition (P) and is pullback flat.

Let us show that S is a right perfect monoid. Clearly, S satisfies condition (K). Suppose that

$$b_1 S \subseteq b_2 S \subseteq b_3 S \subseteq \ldots$$

is an ascending chain of cyclic subacts of a right S-act B_S. Then there exist $s, t \in S$ such that $b_1 = b_2 s$ and $b_2 = b_3 t$. If $s = 1$ or $t = 1$ then we are done. If $s = t = 0$ then $b_1 = b_2 0 = (b_3 0)0 = b_3 0 = b_2$. Hence S satisfies condition (A), too. Thus S is a right perfect monoid, which means that A_S is also projective. ∎

3.8 Freeness

Here we try to answer the questions "When are all right Rees factor acts with property X free?".

Proposition 3.59 ([20]) *All pullback flat right Rees factor acts of S are free if and only if S is not left collapsible or $S = \{1\}$.*

Proposition 3.59 and Corollary 3.50 imply the following result.

Corollary 3.60 *All weakly pullback flat right Rees factor acts of S are free if and only if S is not a right reversible weakly left collapsible monoid or $S = \{1\}$.*

Corollary 3.61 *The following assertions are equivalent for a monoid S:*

1. *All weakly homoflat Rees factor acts of S are free.*
2. *All right Rees factor acts of S satisfying condition* (P) *are free.*
3. *S is not right reversible or $S = \{1\}$.*

Proof. Obviously 1. $\Rightarrow$ 2. The equivalence of conditions 2. and 3. was proved in [20]. Implication 3. $\Rightarrow$ 1. follows from Proposition 3.26 by using 2. ∎

Using Corollaries 3.53 and 3.61 we obtain the following corollary.

Corollary 3.62 *The following assertions are equivalent for a monoid S:*

1. *All principally weakly homoflat right Rees factor acts of S are free.*
2. *All principally weakly flat right Rees factor acts of S are free.*
3. *All torsion free right Rees factor acts of S are free.*
4. *All right Rees factor acts of S are free.*
5. *$S = \{1\}$.*

Note that equivalence of conditions 2, 3, 4 and 5 of this corollary was proved in [20].

To prove the following corollary we need a proposition.

Proposition 3.63 **([24])** *All pullback flat right S-acts are free if and only if S is a group.*

This proposition together with Corollary 3.60 (note that a group is both right reversible and weakly left collapsible) imply the following result for arbitrary acts.

Corollary 3.64 *The following assertions are equivalent for a monoid S:*

1. *All right S-acts are free.*
2. *All torsion free right S-acts are free.*
3. *All principally weakly flat right S-acts are free.*
4. *All principally weakly homoflat right S-acts are free.*
5. *All weakly homoflat right S-acts are free.*
6. *All right S-acts satisfying condition* (P) *are free.*
7. *All weakly pullback flat right S-acts are free.*
8. $S = \{1\}$.

3.9 Synopsis

Finally we present our results in the form of tables. We tabulate our results obtained for Rees factor acts and arbitrary acts and for Rees factor acts, cyclic acts and arbitrary acts over idempotent monoids.

Rows and columns of tables are labelled with flatness properties of acts, the abbreviations used are the same as for Scheme 2. These properties are arranged in order of decreasing strength. In the cell at the intersection of row labelled X and column labelled Y is the class of all monoids such that all right acts (or cyclic right acts or right Rees factor acts) with property X over them have property Y. Since any property in any table implies the property to the right of it or below it, we see that any class of monoids is contained in every class lying above it or to the right of it. Actually the class of monoids in a cell is the intersection of the classes above it and to the right of it. So the diagonal cells play a crucial role, because if we knew them we could, in principle, fill the whole table.

For the tables we have used the following abbreviations:

(*)	— no nontrivial right ideal is left stabilizing
(**)	— no nontrivial right ideal is left stabilizing and left annihilating
(***)	— no nontrivial right ideal is left stabilizing and strongly left annihilating
l.s.	— left stabilizing
l.ann.	— left annihilating
str.l.ann.	— strongly left annihilating
$\exists$l.zero	— S has a left zero
$\exists$zero	— S has a zero
l.coll.	— left collapsible
wlc	— weakly left collapsible
r.rev.	— right reversible
C	— right cancellative
C^0	— right cancellative with a zero adjoined
G	— group
G^0	— group with a zero adjoined
LAR	— left almost regular
Reg.	— regular.

Since idempotent monoids are regular, all acts over idempotent monoids are principally weakly flat by [16]. In addition, pullback flatness and condition (P) are the same for every act over an idempotent monoid by [1].

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KONSERVATIIVSED RUUDUD JA POLÜGOONIDE LAMEDUSEGA SEOTUD OMADUSED

Kokkuvõte

Selles töös vaadeldakse polügoone üle monoidi (ehk S-polügoone, kus S on monoid) ja polügoonide neid omadusi, mis on ühel või teisel viisil seotud lamedusega. Täpsemalt öeldes käsitletakse monoidide homoloogilist klassifikatsiooni polügoonide omaduste järgi üle nende monoidide. See tähendab, et vastust otsitakse järgmist tüüpi küsimustele: "Milliseid tingimusi peab rahuldama monoid, et kõik mingi omadusega parempoolsed polügoonid üle selle monoidi oleksid ka mingi teise omadusega?". Levinumad lamedusega seotud omadused, mida on homoloogilise klassifikatsiooni käigus siiani vaadeldud, on tugev lamedus, tingimus (P), lamedus, nõrk lamedus, spetsiaalne nõrk lamedus ja väändetus.

Parempoolset S-polügooni nimetatakse tugevalt lamedaks, kui tensorkorrutamine temaga säilitab kõik konservatiivsed ruudud vasakpoolsete S-polügoonide kategoorias. 2. peatükis defineeritakse seda säilitamise nõuet formaalselt nõrgendades rida omadusi, mis järelduvad tugevast lamedusest. Edasi uuritakse, millised neist omadustest defineerivad erinevad polügoonide klassid. Tõestatakse, et selliselt üldistades saame kätte kõik eespoolmainitud lamedusega seotud omadused ning et lisaks sellele tekib veel 3 uut polügoonide klassi.

3. peatükis vaadeldakse monoidide homoloogilist klassifikatsiooni, kusjuures erilise tähelepanu all on 'uued' omadused. Leitakse, millal Reesi faktorpolügoonide korral ühest vaatluse all olevast omadusest järeldub teine, real juhtudel on leitud vastus ka tsükliliste või suvaliste polügoonide jaoks. Näiteks on juba varemuuritud omaduste jaoks leitud vastus küsimusele, milliste monoidide korral on kõik väändeta parempoolsed polügoonid spetsiaalselt nõrgalt lamedad. Eraldi on käsitletud idempotentseid monoide. Homoloogilise klassifikatsiooniga seotud tulemused on koondatud 3. peatüki lõpus olevatesse tabelitesse.

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Teadustegevus

On uurinud monoidide homoloogilist klassifikatsiooni polügoonide omaduste järgi üle nende.

On esinenud ettekandega poolrühmade teooria konverentsidel Prahas (1996) ja Tartus (1996).

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