Principal Varieties of Finite Congruences

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Equations have been a fundamental tool in mathematics for centuries, especially in algebra. The classical example in universal algebra is Birkhoff's theorem that establishes the connection between equational theories and varieties of algebras. A related result of Eilenberg and Schützenberger (1976) connects sequences of equations and *varieties of finite algebras* (VFA) which are often called also pseudovarieties. Let us note that in this presentation we are interested only in algebras having a given finite signature Σ.

The VFAs have later been characterized by certain families of finite congruences of term algebras called *varieties of finite congruences* (VFC). In a VFC $\Gamma = \{\Gamma(X)\}_X$, the index X ranges over all finite alphabets, called *leaf alphabets* and used as generating sets of term algebras, and for each X, $\Gamma(X)$ is a filter of finite congruences on the term algebra $T_\Sigma(X)$; a congruence is *finite* if it has a finite number of classes. Moreover, there is a certain connection between $\Gamma(X)$ and $\Gamma(Y)$ for any X and Y.

Our main topic is an interesting special subclass of VFCs, the *principal varieties of finite congruences* (pVFC). For each X, the filter $\Gamma(X)$ of a pVFC Γ is *principal*, i.e., generated by a single congruence. This special class was introduced already by Steinby (1992) along with some general observations concerning VFCs, but it was not investigated any further.

Studying pVFCs is motivated from the algebraic point of view by the fact that all VFCs are decomposable into unions of *finitely generated VFCs*, and all finitely generated VFCs are also principal. A natural first question is whether all pVFCs are finitely generated, and this turns out not to be the case. Therefore the finitely generated VFCs are the very basic building blocks of VFCs, but the principal VFCs form an interesting superclass.

The structure of a pVFC is in general slightly more complicated than the structure of a finitely generated VFC, but we still retain the finiteness of all congruence lattices, which is an important feature from the viewpoint of applications. In fact, applications in tree language and automata theory are the second reason why principal varieties are especially appealing. For each VFA, and hence also for each VFC, there is a unique indexed family of tree languages, a *variety of tree languages* (VTL), which corresponds to these classes bijectively. This three-way connection has proven useful when defining concrete classes of tree languages. For many concrete tree language classes like definite, locally testable or piecewise testable tree languages, the corresponding class of congruences has natural and relatively simple properties, and congruences can be very useful also for further characterizations of the tree language classes. Often the easiest way to define a VTL is to use principal varieties of finite congruences which may or may not be finitely generated.

In addition to the elementary concepts related to principal varieties of finite congruences we investigate closer the general structural relations between the filters of a pVFCs. We answer for example the following question: If we know finitely many of the filters constituting a principal variety of finite congruences, how much do we know about the rest of the variety? Our result determines an interval of pVFCs where our partially known variety must be included. Moreover, each variety in this interval could be a possible solution. This interval is actually strongly connected to certain sets of identities determined by the corresponding VFA, which is not surprising, since each filter of a pVFC is generated by a fully invariant congruence on a term algebra, and one of Birkhoff's original results was that fully invariant congruences have a natural interpretation as a deductively closed class of identities. One further consequence of these investigations is that if we know only finitely many filters of a given VFC, we cannot in general say even whether it is principal or not.

We present also a bijective correspondence between the principal varieties of finite congruences and locally finite varieties of algebras, and as a consequence the finitely generated VFCs and the finitely generated varieties of algebras will correspond to each other bijectively as well. This result provides an interesting link from tree language theory to a widely researched subject in universal algebra. As an application of the theory we demonstrate how to construct a concrete tree language variety using principal varieties.

At the end of the talk we will give some pointers to related topics which might have some interesting connections to the presented results. Some topics that are closely related to pVFCs include for example equational logic, locally finite varieties, and the structural theory of finite algebras.