Composing directed containers: distributive law based composition of monoid-like comonads is like the Zappa-Szép product

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Containers [1] are an elegant representation of a wide class of datatypes in terms of shapes and positions in shapes. Directed containers [2] are a special case accounting for the common situation where every position in a shape determines another shape, intuitively a subshape rooted by that position, some examples being nonempty lists, node-labelled trees, zippers. While containers interpret into set functors via a fully faithful functor, directed containers interpret into comonads. In fact, the category of directed containers is isomorphic to that of comonoids in the category of containers (with respect to the composition monoidal structure), which in turn is easily seen to be the pullback of the interpretation functor of containers into set functors and the forgetful functor from the category of comonads.

The structure of a directed container is intricate because it involves dependent typing, but resembles a monoid and degenerates to one in the case of a single shape.

A sufficient condition for the composition of the underlying functors of two comonads to carry a comonad structure is that the two comonads distribute over each other, i.e., there is a natural transformation between the two functors satisfying certain equations. We show that distributive laws between interpretations of two directed containers are in bijective correspondence with what we call distributive laws between these directed containers. Remarkably, the operations and equations of a distributive law between two directed containers turn out to generalize the structure defining the Zappa-Szép product of two monoids (viz. two mutual actions) [3].

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References

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