Remarks on the star ordering on Rickart *-rings

Jānis Cīrulis
University of Latvia
Riga, Latvia

An involution ring $R := (R, +, \cdot, 0, ^*)$ is called a Rickart ring if, for every $a \in R$, the right annihilator of a is a principal right ideal generated by a (unique) projection a'. We assume the *-cancellation law: if $x^*x = 0$, then $x = 0$. The star order on R is defined by $x \preceq y$ iff $xx^* = x^*y$ and $x^*x = yx^*$ [1]. It was proved in [2] that every initial segment $[0, x]$ of R under this order is isomorphic to an orthomodular lattice, and an immediate conseqence was made that then the join of any pair of elements a and b exists in R whenever the pair is bounded, i.e., $a, b \preceq x$ for some x.

In fact, such a consequence cannot be drawn from the mentioned isomorphism theorem: a local join in $[0, x]$ need not be a global join in R. We show by a direct proof that the consequence is nevertheless true. Moreover, we present explicit equational descriptions of joins and meets of bounded pairs of elements in terms of the ring operations and $'$. We also show that the new operations extend the usual lattice operations for projections, and give a simple sufficient and necessary condition for existence of an upper bound for two elements.

References

[1] M.P. Drazin, Natural structures on semigroups with involution, *Bull. Amer. Math. Soc.* 84, 1978, 139–141.

[2] M.F. Janowitz, On the $*$-order for Rickart *-rings, *Algebra Univers.* 16, 1983, 360–369.