Fooling-sets and rank in nonzero characteristic

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An $n \times n$ matrix M over some field $\mathbf{K}$ is called a *fooling-set matrix of size* n if its diagonal entries are all nonzero, but for all $k \neq \ell$, we have $M_{k,\ell} M_{\ell,k} = 0$. The term "fooling-set" originates from Communication Complexity.

Dietzfelbinger, Hromkovič, & Schnitger (1996) proved that the rank of a fooling-set matrix of size n is at most $\sqrt{n}$, i.e., $n \leq (\mathrm{rk}_{\mathbf{K}} M)^2$, and asked, whether the exponent on the rank in the right-hand side of the inequality can be improved or not.

We settle the question for fields $\mathbf{K}$ of nonzero characteristic.

In a departure from earlier attempts to give lower bounds for the exponent on the rank, we use linear recurring sequences. For every prime p and positive integer t, we construct a sequence in $\mathbf{F}_p$, which gives us a fooling-set matrix $M = M(t)$ of size $n = n(t)$ and rank $r = r(t)$, with $n(t) \to \infty$ as $t \to \infty$, and $n(t)/r(t)^2 \to 1$. While the rank is easy, verifying the fooling-set property for the off-diagonal entries requires to first determine the period of the sequence.

References

[1] Martin Dietzfelbinger, Juraj Hromkovič, and Georg Schnitger, *A comparison of two lower-bound methods for communication complexity*, Theoret. Comput. Sci. **168** (1996), no. 1, 39–51, 19th International Symposium on Mathematical Foundations of Computer Science (Košice, 1994).