ON MONOMORPHISMS AND EPIMORPHISMS IN VARIETIES OF ORDERED ALGEBRAS

VALDIS LAAN AND SOHAIL NASIR

Dedicated to Sydney Bulman-Fleming

Abstract. We study morphisms in varieties of ordered universal algebras. We prove that (i) monomorphisms are precisely the injective homomorphisms, (ii) every regular monomorphism is an order embedding, but the converse is not true in general. Also, we give a necessary and sufficient condition for a morphism to be a regular epimorphism. Finally, we discuss factorizations in such varieties.

1. Preliminaries

Every variety of ordered universal algebras (as defined in [3] or [7]) is a category. It is natural to ask, what meaning have the basic categorical notions in these categories. In this article we ask: what are the (regular) epi- or monomorphisms in such varieties? It turns out that, although, for example, monomorphisms and regular monomorphisms coincide in the varieties of unordered algebras (**reference**), this is not the case in the ordered situation. After describing several types of mono- and epimorphisms we finally show that each variety of ordered algebras has two kinds of factorization systems.

As usual in universal algebra, the *type* of an algebra is a (possibly empty) set Ω which is a disjoint union of sets Ω_k, $k \in \mathbb{N} \cup \{0\}$.

Definition 1 ([3])**.** Let Ω be an ordered type. An *ordered* Ω*-algebra* (or simply ordered algebra) is a triplet $\mathcal{A} = (A, \Omega_A, \leq_A)$ comprising a poset $(A, \leq -A)$ and a set Ω_A of operations on A (for every k-ary operation symbol $\omega \in \Omega_k$ there is a k-ary operation $\omega_A \in \Omega_A$ on A) such that all the operations ω_A are monotone mappings, where monotonicity of ω_A ($\omega \in \Omega_k$) means that

$$a_1 \leq_A a_1' \wedge \ldots \wedge a_k \leq_A a_k' \Longrightarrow \omega_A(a_1, \ldots, a_k) \leq_A \omega_A(a_1', \ldots, a_k')$$

for all $a_1, \ldots, a_k, a_1', \ldots, a_k' \in A$.

A *homomorphism* $f : \mathcal{A} \longrightarrow \mathcal{B}$ of ordered algebras is a monotone operation-preserving map from an ordered Ω-algebra $\mathcal{A}$ to an ordered Ω-algebra $\mathcal{B}$. A *subalgebra* of an ordered algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$ is a subset B of A, which is closed

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under operations and equipped with the order $\leq_B \; = \; \leq_A \cap (B \times B)$. On the direct product of ordered algebras the order is defined componentwise.

If θ is a preorder on a poset $(A, \leq)$ and $a, a' \in A$ then we write

$$a \underset{\theta}{\leq} a' \iff (\exists n \in \mathbb{N})(\exists a_1, \dots, a_n \in A)(a \leq a_1 \theta a_2 \leq a_3 \theta \dots \theta a_n \leq a').$$

An *order-congruence* on an ordered algebra $\mathcal{A}$ is an algebraic congruence θ such that the following condition (cf. [6]) is satisfied,

$$(\forall a, a' \in A) \left(a \underset{\theta}{\leq} a' \underset{\theta}{\leq} a \Longrightarrow a \theta a' \right).$$

Besides order-congruences, also admissible preorders play an important role in the theory of ordered algebras. A preorder ρ on an ordered algebra $\mathcal{A}$ is called *admissible* (see [3]) if it is compatible with operations and extends the order of $\mathcal{A}$. (In [7] such relations are called *quasiorders*.)

Definition 2. A mapping $f : \mathcal{A} \to \mathcal{B}$ between posets $\mathcal{A} = (A, \leq_A)$ and $\mathcal{B} = (B, \leq_B)$ is called an *order embedding* if

$$a \leq_A a' \iff f(a) \leq_B f(a')$$

for all $a, a' \in A$.

Definition 3 ([7])**.** A homomorphism $g : \mathcal{A} \to \mathcal{B}$ of ordered Ω-algebras is called a *Q-homomorphism* if, for all $b, b' \in B$, $b \leq_B b'$ implies that there exist $a_1, a_1', a_2, a_2', \dots, a_n, a_n' \in \mathcal{A}$ such that

$$\begin{array}{ccccccc} b = g(a_1) & & g(a_1') = g(a_2) & \cdots & & g(a_n') = b' \\ & a_1 \leq_A a_1' & & & a_n \leq_A a_n' & \end{array} \tag{1.1}$$

or, shortly, there exist $a, a' \in A$ such that

$$b = g(a), \; a \underset{\ker g}{\leq} a', \; g(a') = b'.$$

We assume that the reader is familiar with basic categorical notions, such as (regular) monomorphism, (regular) epimorphism, equalizer and coequalizer (the definitions can be found, e.g., in [1]).

Definition 4. (Def. 7.74 of [1]) An epimorphism $g : A \longrightarrow C$ in a category $\mathcal{C}$ is called an *extremal epimorphism* if $g = f \circ h$, with $h : A \longrightarrow B$ being a morphism and $f : B \longrightarrow C$ a monomorphism in $\mathcal{C}$, implies that f is an isomorphism.

Definition 5. (Def. 7.61 of [1]) A monomorphism $g : A \longrightarrow C$ in a category $\mathcal{C}$ is called an *extremal monomorphism* if $g = h \circ f$, with $f : A \longrightarrow B$ being an epimorphism and $h : B \longrightarrow C$ a morphism in $\mathcal{C}$, implies that f is an isomorphism.

From Corollary 7.63 and Proposition 7.75 of [1] we have the following fact.

Proposition 1. *Every regular monomorphism (resp. epimorphism) is an extremal monomorphism (resp. epimorphism).*

All ordered Ω-algebras and their homomorphisms form a category. Considering the set of homomorphisms from $\mathcal{A}$ to $\mathcal{B}$ (for each $\mathcal{A}$ and $\mathcal{B}$) as a poset with respect to pointwise order we may regard the category of all ordered Ω-algebras as a Pos-category (a category enriched over the category Pos of posets and monotone mappings), because this pointwise order is compaticle with composition of homomorphisms. In a Pos-category one may study some more types of monomorphisms and epimorphisms.

Definition 6. We say that a morphism f in a Pos-category $\mathcal{C}$ is a *submonomorphism* if $f \circ u \leq f \circ v$ implies $u \leq v$ for all morphisms u, v.

Obviously, every submonomorphism is a monomorphism.

Definition 7. We call an epimorphism f in a Pos-category $\mathcal{C}$ a *subextremal epimorphism* if $f = m \circ e$, where m is a submonomorphism, implies that m is an isomorphism.

Every extremal epimorphism is a subextremal epimorphism.
Dually we define subepimorphisms and subextremal monomorphisms.

Definition 8. We say that a pair (B, f) is a *subcoequalizer* of a pair (u, v) of morphisms $C \to A$ in a Pos-category $\mathcal{C}$ if $f : A \to B$ is a morphism such that

(1) $f \circ u \leq f \circ v$;
(2) if $f' : A \to B'$ is a morphism in $\mathcal{C}$ such that $f' \circ u \leq f' \circ v$ then there exists a unique morphism $h : B \to B'$ such that $h \circ f = f'$.

Subcoequalizers are a special case of a general notion of 2-categoical limits called coinserters in **Kelly?**. In [4], subcoequalizers are considered in the category of right S-posets over a pomonoid S (these are also ordered algebras).

Definition 9. We call an epimorphism f in a Pos-category $\mathcal{C}$ a *subregular epimorphism* if it is a subcoequalizer of a pair of morphisms in $\mathcal{C}$.

Lemma 1. *Every subregular epimorphism is a subextremal epimorphism.*

Proof. Suppose that an epimorphism $f : A \to B$ is a subcoequaliser of $u, v : C \to A$ and f factorizes as $f = m \circ e$, where $m : D \to B$ is a submonomorphism. Then $m \circ e \circ u = f \circ u \leq f \circ v = m \circ e \circ v$. Since m is a submonomorphism, $e \circ u \leq e \circ v$. Hence there exists a unique $h : B \to D$ such that $h \circ f = e$. Now $m \circ h \circ f = m \circ e = f$ implies $m \circ h = 1_B$, because f is an epimorphism, and $m \circ h \circ m = 1_B \circ m = m$ implies $h \circ m = 1_D$, because m is a monomorphism. Thus m is an isomorphism. $\square$

2. Quotients

There are two natural ways of forming quotients of ordered algebras. If θ is an order-congruence on an ordered Ω-algebra $\mathcal{A} = (A, \Omega_A, \leq_A)$ then one can define an order relation $\preceq$ on the quotient set A/θ by

$$[a] \preceq [a'] \iff a \underset{\theta}{\leq} a',$$

$a, a' \in A$. With the natural definitions of operations we obtain an ordered Ω-algebra $\mathcal{A}/\theta := \big(A/\theta, \Omega_{A/\theta}, \preceq\big)$. We call such an algebra a *regular quotient algebra* of $\mathcal{A}$ by an order-congruence θ, because, as we shall see in Theorem 3, the natural surjection $\theta^\sharp : A \to A/\theta, a \mapsto [a]$ is a regular epimorphism. Such quotients were introduced in [6] (see Proposition 2.1).

The other type of quotient algebras is defined as follows. Let ρ be an admissible preorder on $\mathcal{A}$. Then $\theta = \rho \cap \rho^{-1}$ is an order-congruence on $\mathcal{A}$ and one can define an order relation $\sqsubseteq$ on the quotient set A/θ by

$$[a] \sqsubseteq [a'] \iff a \rho a',$$

$a, a' \in A$. With the natural operations we obtain an ordered algebra $\mathcal{A}/\rho := \big(A/(\rho \cap \rho^{-1}), \Omega_{A/(\rho\cap\rho^{-1})}, \sqsubseteq\big)$, which we call just a *quotient algebra* of $\mathcal{A}$ by an admissible preorder ρ, because the natural mapping $A \to A/(\rho \cap \rho^{-1})$ is just a surjective homomorphism (not necessarily a regular epimorphism). Such quotients appear in [3] and [7].

If ρ is an admissible preorder then we may consider both the regular quotient algebra $\mathcal{A}/(\rho\cap\rho^{-1})$ and the nonregular quotient algebra $\mathcal{A}/\rho$. They have the same elements and operations, but the order may be different: $\preceq$ is contained in $\sqsubseteq$. So the quotient algebra $\mathcal{A}/\rho$ is possibly "more ordered" than the regular quotient algebra $\mathcal{A}/(\rho\cap\rho^{-1})$.

Given a homomorphism $f : \mathcal{A} \longrightarrow \mathcal{B}$ of ordered algebras, the *directed kernel* $\overrightarrow{\ker} f$ *of* f, defined by

$$\overrightarrow{\ker} f = \{(a, a') \in \mathcal{A} \times \mathcal{A} : f(a) \leq f(a')\}$$

(see [7]) is clearly an admissible preorder on $\mathcal{A}$. Hence

$$\ker f = (\overrightarrow{\ker} f) \cap (\overrightarrow{\ker} f)^{-1}$$

is an order-congruence on $\mathcal{A}$. So we have the quotient algebras

$$\mathcal{A}/\ker f = \left(A/\ker f, \Omega_{A/\ker f}, \preceq\right), \quad \mathcal{A}/\overrightarrow{\ker} f = \left(A/\ker f, \Omega_{A/\ker f}, \sqsubseteq\right),$$

where the relations $\preceq$ and $\sqsubseteq$ are defined by

$$\begin{aligned} [a] \preceq [a'] &\iff a \underset{\ker f}{\leq} a', \\ [a] \sqsubseteq [a'] &\iff f(a) \leq f(a') \iff (a, a') \in \overrightarrow{\ker} f. \end{aligned}$$

This observation allows to formulate the Homomorphism Theorem in the following way.

Theorem 1. *For any homomorphism* $f : \mathcal{A} \to \mathcal{B}$ *of ordered* Ω*-algebras the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\quad f \quad} & \mathcal{B} \\ {\scriptstyle \pi}\downarrow & & \uparrow{\scriptstyle \iota} \\ \mathcal{A}/\ker f & \xrightarrow[1_{A/\ker f}]{} & \mathcal{A}/\overrightarrow{\ker} f \end{array}$$

commutes where the mappings π, ι *are defined by*

$$\begin{aligned} \pi(a) &:= [a], \\ \iota([a]) &:= f(a). \end{aligned}$$

Moreover,

(1) π *is a Q-homomorphism;*
(2) $1_{A/\ker f}$ *is injective;*
(3) ι *is an order embedding;*
(4) $1_{A/\ker f} \circ \pi$ *is surjective;*
(5) $\iota \circ 1_{A/\ker f}$ *is injective.*

Proof. This is straightforward. □

It is possible that for a morphism $f : \mathcal{A} \to \mathcal{B}$ the quotients $\mathcal{A}/\ker f$ and $\mathcal{A}/\overrightarrow{\ker} f$ are indeed non-isomorphic.

Example 1. Consider the pomonoids $\mathcal{A} = (\mathbb{N}, \cdot, \leq_2)$ and $\mathcal{B} = (\mathbb{N}, \cdot, \leq)$ where $\leq$ is the usual order of natural numbers and $\leq_2$ is defined by

$$m \leq_2 n \iff m, n \in 2\mathbb{N} \text{ and } m \leq n.$$

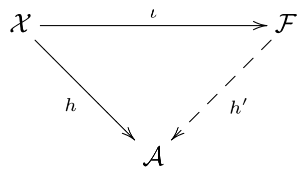
Then $f = 1_{\mathbb{N}} : \mathcal{A} \to \mathcal{B}$ is clearly a pomonoid homomorphism, for which $\ker f = \Delta_{\mathbb{N}}$, $\overrightarrow{\ker} f = \leq$ and $\underset{\ker f}{\leq_2} = \leq_2$. Hence $\mathcal{A}/\ker f \cong \mathcal{A}$, but $\mathcal{A}/\overrightarrow{\ker} f \cong \mathcal{B}$.

3. Free ordered algebras and varieties

Let us recall the definition of free ordered algebras over a poset.

Definition 10 (Cf. [7], Def. 2.1)**.** Let $\mathcal{K}$ be a class of ordered Ω-algebras, $\mathcal{X} = (X; \leq)$ a poset, $\mathcal{F} \in \mathcal{K}$, and let $\iota : X \to F$ be an order embedding. $\mathcal{F}$ is called the *free Ω-algebra over $\mathcal{X}$ in $\mathcal{K}$* with the canonical mapping ι, if the following hold:

(1) $\iota(X)$ generates $\mathcal{F}$;
(2) given any monotone mapping $h : \mathcal{X} \to \mathcal{A}$, where $\mathcal{A} \in \mathcal{K}$, there exists a (unique) homomorphism $h' : \mathcal{F} \to \mathcal{A}$ such that $h' \circ \iota = h$.



Example 2. In [5] it is shown that free objects in the category of right S-posets (S is a pomonoid) exist.

From [7], Theorem 2.4, we have the following result.

Theorem 2. *Let $\mathcal{X}$ be a poset. If $\mathcal{K}$ is a class of ordered algebras which is closed under isomorphisms, subalgebras and products, then the free algebra on $\mathcal{X}$ exists in $\mathcal{K}$ provided that $\mathcal{K}$ contains an at least two-element algebra.*

We are interested in the case where the class $\mathcal{K}$ is a variety.

Definition 11 ([3])**.** A class of ordered Ω-algebras is called a **variety**, if it is closed under isomorphisms, quotients, subalgebras and products.

An **inequality** of type Ω is a sequence of symbols $t \leq t'$, where t, t' are Ω-terms. We say that "$t \leq t'$ holds in an ordered algebra $\mathcal{A}$" if $t_{\mathcal{A}} \leq t'_{\mathcal{A}}$ where $t_{\mathcal{A}}, t'_{\mathcal{A}} : A^n \to A$ are the functions on $\mathcal{A}$ induced by t and t'. Of course, inequalities $t \leq t'$ and $t' \leq t$ hold if and only if the identity $t = t'$ holds. From [3] we have a Birkhoff-type characterization for varieties: a class $\mathcal{K}$ of Ω-algebras is a variety if and only if it consists precisely of all the algebras satisfying some set of inequalities.

Example 3. Lattices, bounded posets, posemigroups or pomonoids form a variety. If S is a pomonoid then the class of all right S-posets is a variety of ordered Ω-algebras, where $\Omega = \Omega_1 = \{\cdot s \mid s \in S\}$, defined by the following set of identities and inequalities:

$$\{(x \cdot s) \cdot t = x \cdot (st) \mid s, t \in S\} \cup \{x \cdot 1 = x\} \cup \{x \cdot s \leq x \cdot t \mid s, t \in S, s \leq t\}.$$

It is easy to see that in addition to (nonregular) quotients, each variety is also closed under regular quotients.

Every variety of ordered Ω-algebras together with their homomorphisms forms a category. Our wish is to study special morphisms in such categories. We start with a description of isomorphisms.

Proposition 2. *A homomorphism in a variety of ordered Ω-algebras is an isomorphism if and only if it is a surjective order embedding.*

Proof. **Necessity.** If $f : \mathcal{A} \to \mathcal{B}$ is an isomorphism then it has an inverse mapping $f^{-1} : \mathcal{B} \to \mathcal{A}$, and hence is surjective. If $f(a) \leq f(a')$, $a, a' \in A$, then $a = (f^{-1} \circ f)(a) \leq (f^{-1} \circ f)(a') = a'$, so f is an order embedding.

Sufficiency. Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism which is a surjective order embedding. Then it has a monotone inverse $f^{-1} : \mathcal{B} \to \mathcal{A}$. It is easy to see that f^{-1} preserves operations. □

4. Monomorphisms

In this section we study different types of monomorphisms in varieties of ordered algebras. We start with monomorphisms and submonomorphisms.

Proposition 3. *Let $\mathcal{V}$ be a variety of ordered Ω-algebras. Then*

(1) *monomorphisms in the category $\mathcal{V}$ are precisely the injective homomorphisms;*
(2) *submonomorphisms in the category $\mathcal{V}$ are precisely the order embeddings.*

Proof. (1) Every injective homomorphism in a concrete category is a monomorphism (**reference!**). For the converse we note that if all algebras in $\mathcal{V}$ have at most one element then between any two objects in $\mathcal{V}$ there is at most one morphism, and hence all morphisms are both monomorphisms and injective. Now suppose that $\mathcal{V}$ has at least two-element algebra and $f : \mathcal{A} \longrightarrow \mathcal{B}$ is a monomorphism in $\mathcal{V}$ such that $f(a) = f(a')$ for some $a, a' \in A$. By Theorem 2 there is a free algebra $\mathcal{F}$ in $\mathcal{V}$ on a (trivially ordered) one-element poset $X = \{x\}$ and an order embedding $\iota : X \to \mathcal{F}$. Hence, for every $c \in A$ there exists a homomorphism $g_c : \mathcal{F} \to \mathcal{A}$ of ordered Ω-algebras such that $g_c(\iota(x)) = c$. Therefore

$$(f \circ g_a)(\iota(x)) = f(a) = f(a') = (f \circ g_{a'})(\iota(x)).$$

Since $\iota(x)$ generates $\mathcal{F}$, we have $f \circ g_a = f \circ g_{a'}$. But then, because f is left cancellable, one has $g_a = g_{a'}$. Consequently,

$$a = g_a(\iota(x)) = g_{a'}(\iota(x)) = a'.$$

(2) Suppose that $f : \mathcal{A} \to \mathcal{B}$ is an order embedding in $\mathcal{V}$ and $f \circ u \leq f \circ v$ for some $u, v : \mathcal{C} \to \mathcal{A}$. Then, for each $c \in C$, $f(u(c)) \leq f(v(c))$. Since f is an order embedding, $u(c) \leq v(c)$. Hence $u \leq v$ and f is a submonomorphism. The converse can be proved very similarly to the case (1). □

Proposition 4. *Every regular (or extremal) monomorphism in a variety of ordered Ω-algebras is an order embedding.*

Proof. Recall that every regular monomorphism is extremal by Proposition 1. Let now $f : \mathcal{A} \to \mathcal{B}$ be an extremal monomorphism where $\mathcal{A} = (A, \Omega_A, \leq_A)$ and $\mathcal{B} = (B, \Omega_B, \leq_B)$ are two algebras in a variety $\mathcal{V}$. Consider also the ordered Ω-algebra $\mathcal{A}' = (A, \Omega_A, \overrightarrow{\ker} f)$. Note that the relation $\overrightarrow{\ker} f$ is antisymmetric because f

is a monomorphism and hence injective by Proposition 3. Let us show that $\mathcal{A}' \in \mathcal{V}$. For this, take any defining inequality $t \leq t'$ of the variety $\mathcal{V}$, where t, t' are Ω-terms in variables $x_1, \ldots, x_n$. Since $t_B \leq t'_B$, for any $a_1, \ldots, a_n \in A$,

$$\begin{aligned} f(t_A(a_1, \ldots, a_n)) &= t_B(f(a_1), \ldots, f(a_n)) \leq_B t'_B(f(a_1), \ldots, f(a_n)) \\ &= f(t'_A(a_1, \ldots, a_n)), \end{aligned}$$

so $t_A(a_1, \ldots, a_n)\ (\overrightarrow{\ker} f)\ t'_A(a_1, \ldots, a_n)$. Thus, indeed, $\mathcal{A}' \in \mathcal{V}$

Now $1_A : \mathcal{A} \longrightarrow \mathcal{A}'$ and $f : \mathcal{A}' \longrightarrow \mathcal{B}$ are homomorphisms of ordered Ω-algebras with $f = f \circ 1_A$. Because 1_A is obviously an epimorphism and f is an extremal monomorphism, 1_A must be an isomorphism. By Proposition 2, 1_A is an order embedding, but this implies immediately that f is an order-embedding. □

There exist varieties of ordered algebras where order embeddings are not necessarily regular monomorphisms.

Example 4. We shall construct a class of order embeddings in the variety of posemigroups which are not regular monomorphisms.

Let $(S, +, \leq)$ be any commutative posemigroup with no idempotents and with the property that $a+c \leq b+c$ implies $a \leq b$, for all $a, b, c \in S$ (for example $(\mathbb{N}, +, \leq)$ and $(\mathbb{R}^+, +, \leq)$ are such). Consider the set

$$T := (S \times S)/\sim = \{[a, b] \mid a, b \in S\},$$

where

$$(a, b) \sim (c, d) \Longleftrightarrow a + d = b + c,$$

the equivalence class of (a, b) is denoted by $[a, b]$, the addition on T is given by

$$[a, b] + [c, d] := [a + c, b + d],$$

and the order by

$$[a, b] \leq [c, d] \Longleftrightarrow a + d \leq b + c.$$

Then T is a commutative group where the addition is monotone. Choose some element $z \in S$. Then the zero element of T is $[z, z] =: 0$, $-[a, b] = [b, a]$ and the mapping $f : S \to T$,

$$f(a) := [a + z, z],$$

$a \in S$, is an order embedding. Indeed, for $a, b \in S$,

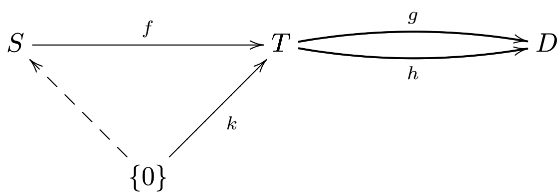
$$[a + z, z] \leq [b + z, z] \Longleftrightarrow a + z + z \leq z + b + z \Longleftrightarrow a \leq b.$$

Let us show that if $u, v : T \to C$ are posemigroup homomorphisms such that $u \circ f = v \circ f$ then $u(0) = v(0)$. Note that $u \circ f = v \circ f$ means that $u([a + z, z]) =$

$v([a+z,z])$ for each $a \in S$. Now we may calculate

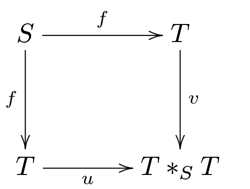
$$\begin{aligned}
u(0) &= u([z,z]) = u([z+z,z]+[z,z+z]) = u([z+z,z]) + u([z,z+z]) \\
&= v([z+z,z]) + u([z,z+z]) \\
&= v([z+z,z]+[z,z+z]+[z+z,z]) + u([z,z+z]) \\
&= v(0) + v([z+z,z]) + u([z,z+z]) \\
&= v(0) + u([z+z,z]) + u([z,z+z]) \\
&= v(0) + u([z+z,z]+[z,z+z]) = v(0) + u(0) \\
&= v([z,z+z]+[z+z,z]) + u(0) \\
&= v([z,z+z]) + v([z+z,z]) + u(0) \\
&= v([z,z+z]) + u([z+z,z]) + u([z,z+z]+[z+z,z]) \\
&= v([z,z+z]) + u([z+z,z]+[z,z+z]+[z+z,z]) \\
&= v([z,z+z]) + u([z+z,z]) = v([z,z+z]) + v([z+z,z]) \\
&= v(0).
\end{aligned}$$

Suppose that f is an equalizer of posemigroup homomorphisms $g, h : T \to D$. Consider the mapping $k : \{0\} \to T, 0 \mapsto 0$, which clearly is a posemigroup homomorphism. Since S has no idempotents, there is no morphism $\{0\} \to S$. Hence, to arrive at contradiction it suffices to prove that $g \circ k = h \circ k$. (**Arrows!**)



From (**Nasir, reference**) we know that the category of pomonoids has pushouts. From this fact it is not difficult to deduce that also the category of posemigroups has pushouts. So let

(4.1)

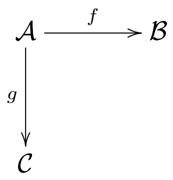


be a pushout diagram. By the above argument we have $u(0) = v(0)$. Since (4.1) is a pushout and $g \circ f = h \circ f$, there exists a posemigroup homomorphism $m : T *_S T \to S$ with $m \circ u = g$ and $m \circ v = h$. But $u(0) = v(0)$ implies $g(0) = h(0)$, and hence $g \circ k = h \circ k$. This gives a contradiction.

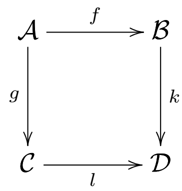
As a particular case of this construction we see that the inclusion $\mathbb{N} \to \mathbb{Z}$ of additive semigroups is an order embedding which is not a regular monomorphism.

However, there are varieties where the converse of Proposition 4 is also true.

We say that a variety $\mathcal{V}$ has *the strong amalgamation property* with respect to class $\mathcal{M}$ of monomorphisms in $\mathcal{V}$ (see [9], p. 93) if each span



with $f, g \in \mathcal{M}$ can be completed to a pullback diagram



with k, l in $\mathcal{M}$. If the above condition is satisfied for $f = g$ then $\mathcal{V}$ is said to satisfy *the special amalgamation property.* In [9] one can find a list of categories having the strong amalgamation property.

Proposition 5. *If a variety $\mathcal{V}$ of ordered algebras has the special amalgamation property with respect to the class of order embeddings then regular monomorphisms in $\mathcal{V}$ coincide with order embeddings.*

Proof. It is straightforward from the definition of a pullbacks and equalizers that every order embedding is a regular monomorphism. □

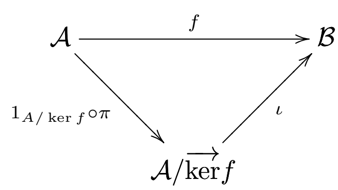
Example 5. Nasir will some examples of varieties with special AP.

5. Epimorphisms

Now we study different types of epimorphisms. We start with subextremal epimorphisms.

Proposition 6. *Every subextremal epimorphism in a variety $\mathcal{V}$ of ordered Ω-algebras is surjective.*

Proof. Let $f : \mathcal{A} \longrightarrow \mathcal{B}$ be a subextremal epimorphism in $\mathcal{V}$. In the commutative diagram (see Theorem 1)



ι is an order embedding, hence a submonomorphism by Proposition 3. Consequently, ι is an isomorphism because f is subextremal. The surjectivity of f now follows from that of $1_{A/\ker f} \circ \pi$. □

Corollary 1. *Every extremal epimorphism in a variety of ordered Ω-algebras is surjective.*

Proof. Every extremal epimorphism is a subextremal epimorphism. □

On the other hand, not every epimorphism is surjective.

Example 6. Interestingly, the same morphism $f : S \to T$, as in Example 4, works if S is not a group. If S is not a group then some equation $x+b = a$, $a, b \in S$, has no solution x in S. Suppose that $f(c) = [a, b]$ for some $c \in S$. Then $[c + z, z] = [a, b]$, i.e. $c + z + b = z + a$ or $c + b = a$, which contradicts the assumption. So f is not surjective.

Let us prove that f is an epimorphism. To this end, let $u \circ f = v \circ f$ for some posemigroup homomorphisms $u, v : T \to C$. Then, as we saw, $u(0) = v(0)$. Hence

$$\begin{aligned}
u([a,b]) &= u([a,b]+0) = u([a,b]) + u(0) = u([a,b]) + v(0) \\
&= u([a,b]) + v([b+z,z] + [z,b+z]) \\
&= u([a+z,z]) + u([z,b+z]) + v([b+z,z]) + v([z,b+z]) \\
&= v([a+z,z]) + u([z,b+z]) + u([b+z,z]) + v([z,b+z]) \\
&= v([a+z,z]) + u(0) + v([z,b+z]) \\
&= v([a+z,z]) + v(0) + v([z,b+z]) \\
&= v([a+z,z] + 0 + [z,b+z]) \\
&= v([a,b])
\end{aligned}$$

for all $a, b \in S$, which means that $u = v$.

Example 7. Let us also point out that order embeddings need not be extremal monomorphisms. Take again the same $f : S \to T$ as in Example 4. It is an order embedding and hence a monomorphism. It factorizes as $f = 1_T \circ f$, where f is an epimorphism as we saw in the previous example. If f were an extremal monomorphism, it would also be an isomorphism. But this is not the case because f is not surjective.

Example 8. Not every surjective epimorphism is extremal. To see this, let $A = \{0, 1\}$ and $B = \{0', 1'\}$ be two copies of the two element semilattice. Endow A with discrete and B with natural order. Then the morphism $g : A \longrightarrow B, 0 \mapsto 0', 1 \mapsto 1'$, is clearly a surjective epimorphism. It factorizes as $g = g \circ 1_A$, where g is a monomorphism. Since g is not an isomorphism, g is not an extremal epimorphism.

Theorem 3. *For an epimorphism $g : \mathcal{A} \longrightarrow \mathcal{B}$ in a variety $\mathcal{V}$ of ordered Ω-algebras, the following are equivalent:*

(1) *g is regular;*
(2) *g is extremal;*
(3) *g is a Q-homomorphism.*

Proof. (1) $\Longrightarrow$ (2) by Proposition 1.

(2) $\Longrightarrow$ (3) Define on B a relation $\preceq$ by

$$b \preceq b' \iff (\exists a, a' \in A)\left(b = g(a) \wedge a \underset{\ker g}{\leq} a' \wedge g(a') = b'\right).$$

We first show that $\preceq$ is a partial order on B.

Because for any $b \in B$ we may write a sequence $b = g(a)$, $a \leq_A a$, $g(a) = b$, the relation $\preceq$ is reflexive. Also from $b \preceq b'$ and $b' \preceq b''$ one may write

$$b = g(a),\ a \underset{\ker g}{\leq} a',\ g(a') = b' = g(a'_1),\ a'_1 \underset{\ker g}{\leq} a'',\ g(a'') = b''$$

whence

$$b = g(a),\ a \underset{\ker g}{\leq} a'',\ g(a'') = b'',$$

or $b \preceq b''$. Thus $\preceq$ is transitive. It is clear that $\preceq$ is contained in $\leq_B$. The anti-symmetry of $\preceq$ now follows from that of $\leq_B$.

We next prove that operations $\omega_B \in \Omega_B$ are monotone with respect to $\preceq$. Suppose, to this end, that ω be an n-ary operation and $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n \in B$ such that

$$b_1 \preceq c_1,\ b_2 \preceq c_2, \ldots,\ b_n \preceq c_n. \tag{5}$$

One may assume that the sequences (of type 1.1) represented by inequalities (5) have all 'length' equal to m (as we can always adjoin segments of type $a \leq a$ and $g(a) = g(a)$ to the shorter sequences). We can therefore write all of them in the following form

$$\begin{array}{ccccccc} b_i = g(a_i) & & g(a_i^1) = g(a_i^2) & \cdots & & g(d_i) = c_i & \\ & a_i \leq_A a_i^1 & & & a_i^m \leq_A d_i & & , \end{array}$$

where for $1 \leq i \leq n$, $1 \leq j \leq m$, $a_i, d_i, a_i^j \in A$. One may now further write a sequence

$$\begin{array}{rcl}
\omega_B(b_1, b_2, \ldots, b_n) & = & \omega_B(g(a_1), g(a_2), \ldots, g(a_n)) = g(\omega_A(a_1, a_2, \ldots, a_n)) \\
\\
\omega_A(a_1, a_2, \ldots, a_n) & \leq_A & \omega_A(a_1^1, a_2^1, \ldots, a_n^1) \\
\\
g(\omega_A(a_1^1, a_2^1, \ldots, a_n^1)) & = & \omega_B(g(a_1^1), g(a_2^1), \ldots, g(a_n^1)) \\
& = & \omega_B(g(a_1^2), g(a_2^2), \ldots, g(a_n^2)) \\
& = & g(\omega_A(a_1^2, a_2^2, \ldots, a_n^2) \\
\\
\omega_A(a_1^2, a_2^2, \ldots, a_n^2) & \leq_A & \omega_A(a_1^3, a_2^3, \ldots, a_n^3) \\
& \vdots & \\
g(\omega_A(d_1, d_2, \ldots, d_n)) & = & \omega_B(g(d_1), g(d_2), \ldots, g(d_n)) = \omega_B(c_1, c_2, \ldots, c_n).
\end{array}$$

Thus $\omega_B(b_1, b_2, \ldots, b_n) \preceq \omega_B(c_1, c_2, \ldots, c_n)$.

We have therefore shown that $(B; \Omega_B; \preceq)$ is an ordered Ω-algebra. We shall henceforth denote it by $\mathcal{B}'$.

The aim will be achieved if we show that the order $\preceq$ coincides with $\leq_B$. Let $g^* : \mathcal{A} \longrightarrow \mathcal{B}'$ be defined by

$$g^*(a) = g(a).$$

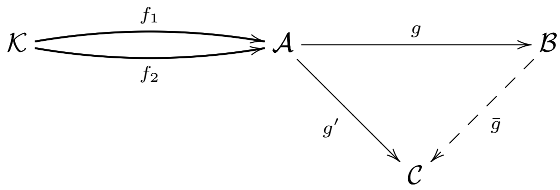
Then g^* preserves the operations because g does. Also, $a \leq_A a'$ clearly implies $g^*(a) \preceq g^*(a')$. Thus $g^* : \mathcal{A} \longrightarrow \mathcal{B}'$ is in fact a homomorphism of ordered Ω-algebras. Now since $\preceq$ is contained in $\leq_B$, by Proposition 3 the identity map $1_B : \mathcal{B}' \longrightarrow \mathcal{B}$ is a monomorphism of ordered Ω-algebras. But then, because g is extremal, $g = 1_B \circ g^*$ implies that 1_B is an isomorphism. The orders $\preceq$ and $\leq_B$ therefore coincide.

(3) $\Longrightarrow$ (1) We need to show that a Q-homomorphism $g : \mathcal{A} \longrightarrow \mathcal{B}$ is the coequalizer of some pair, f_1, f_2 say, of homomorphisms of ordered Ω-algebras.

Take $\mathcal{K}$ to be the kernel of g (which is in fact an ordered subalgebra of $\mathcal{A} \times \mathcal{A}$) and $f_1, f_2 : \mathcal{K} \longrightarrow \mathcal{A}$ to be the restrictions of the projections from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then obviously $g \circ f_1 = g \circ f_2$. Let $g' : \mathcal{A} \longrightarrow \mathcal{C}$ be a homomorphism of ordered Ω-algebras with $g' \circ f_1 = g' \circ f_2$. Define $\bar{g} : \mathcal{B} \longrightarrow \mathcal{C}$ by

$$\bar{g}(b) = g'(a),$$

where $b = g(a)$, $a \in A$ (recall that g is surjective). To see that $\bar{g}$ is well-defined, assume that $g(a_1) = b = g(a_2)$, $a_1, a_2 \in A$. Then $(a_1, a_2) \in \mathcal{K}$, whence $g'(a_1) = (g' \circ f_1)(a_1, a_2) = (g' \circ f_2)(a_1, a_2) = g'(a_2)$.



To prove that $\bar{g}$ is a homomorphism of Ω-algebras take any $\omega \in \Omega_n$ and elements $b_1, \ldots, b_n \in B$. Then since g is surjective there must exist $a_1, \ldots, a_n \in A$ such that $g(a_i) = b_i$, $1 \leq i \leq n$. Now we have

$$\begin{aligned} \bar{g}(\omega_B(b_1, \ldots, b_n)) &= \bar{g}(\omega_B(g(a_1), \ldots, g(a_n))) \\ &= \bar{g}(g(\omega_A(a_1, \ldots, a_n)) \\ &= g'(\omega_A(a_1, \ldots, a_n)) \\ &= \omega_C(g'(a_1), \ldots, g'(a_n)) \\ &= \omega_C(\bar{g}(b_1), \ldots, \bar{g}(b_n)). \end{aligned}$$

Hence $\bar{g}$ is a homomorphism of Ω-algebras.

We now show that $\bar{g}$ preserves the order. To this end, let $b \leq_B b'$. Then there exist, by assumption, $a_1, a_1', a_2, a_2', \ldots, a_n, a_n' \in A$ such that

$$\begin{array}{ccccccc} b = g(a_1) & & g(a_1') = g(a_2) & & \cdots & & g(a_n') = b' \\ & a_1 \leq a_1' & & a_2 \leq a_2' & & a_n \leq a_n' & \end{array} .$$

But then one may simply calculate

$$\begin{aligned} \bar{g}(b) &= \bar{g}(g(a_1)) = g'(a_1) \\ &\leq g'(a_1') = \bar{g}(g(a_1')) = \bar{g}(g(a_2)) \\ &\leq \cdots \\ &\leq g'(a_n') = \bar{g}(g(a_n')) = \bar{g}(b'). \end{aligned}$$

Thus $\bar{g}$ is a homomorphism of ordered Ω-algebras. Moreover, by the definition of $\bar{g}$ it is clear that $g' = \bar{g} \circ g$.

Lastly, it only remains to show that $\bar{g}$ is unique. Suppose $\tilde{g} : \mathcal{B} \longrightarrow \mathcal{C}$ be another homomorphism with $\tilde{g} \circ g = g'$. Then for any $b \in B$, with $g(a) = b$, $a \in A$, we have

$$\tilde{g}(b) = g'(a) = \bar{g}(b).$$

This completes the proof. □

Corollary 2. *Regular and extremal epimorphisms in the varieties of (unordered) Ω-algebras, that coincide by above theorem, are precisely the surjective morphisms.*

Proof. That every regular (equivalently extremal) epimorphism of Ω-algebras is surjective, follows from Proposition 6. Also because there exists (trivially) a sequence of type (1.1) for for any $b = b'$ for $b \in B$, every surjective homomorphism is a regular (equivalently extremal) epimorphism. □

It turns out that surjective epimorphisms also admit a description in categorical terms.

Theorem 4. *For an epimorphism $g : \mathcal{A} \longrightarrow \mathcal{B}$ in a variety $\mathcal{V}$ of ordered Ω-algebras, the following are equivalent:*

(1) *g is subregular;*
(2) *g is subextremal;*
(3) *g is surjective.*

Proof. (1) $\Longrightarrow$ (2) by Lemma 1.

(2) $\Longrightarrow$ (3) by Proposition 6.

(3) $\Longrightarrow$ (1) Let $g : \mathcal{A} \longrightarrow \mathcal{B}$ be a surjective epimorphism in $\mathcal{V}$. Take $\mathcal{K}$ to be the directed kernel $\overrightarrow{\ker} g \subseteq \mathcal{A} \times \mathcal{A}$ with componentwise order and operations and $f_1, f_2 : \mathcal{K} \longrightarrow \mathcal{A}$ to be the restrictions of the projections from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$. Then obviously $g \circ f_1 \leq g \circ f_2$.

Suppose that $g' \circ f_1 \leq g' \circ f_2$ for some $g' : \mathcal{A} \longrightarrow \mathcal{C}$ in $\mathcal{V}$. Define $\bar{g} : \mathcal{B} \longrightarrow \mathcal{C}$ by

$$\bar{g}(b) = g'(a),$$

where $b = g(a)$, $a \in A$. To see that $\bar{g}$ is well-defined, assume $g(a_1) = b = g(a_2)$, $a_1, a_2 \in A$. Then $(a_1, a_2), (a_2, a_1) \in \overrightarrow{\ker} g$, whence $g'(a_1) = (g' \circ f_1)(a_1, a_2) \leq (g' \circ f_2)(a_1, a_2) = g'(a_2)$, and similarly $g'(a_2) \leq g'(a_1)$. Thus $g'(a_1) = g'(a_2)$.

That $\bar{g}$ is a homomorphism of Ω-algebras can be proved in the same way as this was done in Theorem 3. Let us show that $\bar{g}$ is monotone. To this end, let $b \leq_B b'$, where $b = g(a), b' = g(a')$, $a, a' \in A$. Then $(a, a') \in \overrightarrow{\ker} g$ and

$$\bar{g}(b) = g'(a) = (g' \circ f_1)(a, a') \leq (g' \circ f_2)(a, a') = g'(a') = \bar{g}(b').$$

It is clear that $g' = \bar{g} \circ g$, and $\bar{g}$ is unique morphism with this property. This completes the proof. □

The following example illustrates that surjective epimorphisms of ordered Ω-algebras are not necessarily regular (equivalently, extremal).

Example 9. Consider the same morphism g as in Example 8. This $g : A \longrightarrow B$ is clearly surjective. It is however not a regular epimorphism, because one can not write a sequence of type (1.1) for $0' \leq_B 1'$.

However, in the category of lower (or upper) semilattices we have the following result.

Corollary 3. *Regular (extremal) epimorphisms of lower semilattices are the surjective homomorphisms.*

Proof. Let $g : A \to B$ be a surjective homomorphism of lower semilattices and $b \leq_B b'$, $b, b' \in B$. Then $b = g(a)$ and $b' = g(a')$ for some $a, a' \in A$. Then $b \wedge b' = b$, so $b = g(a) = g(a) \wedge g(a') = g(a \wedge a')$, and we have a sequence

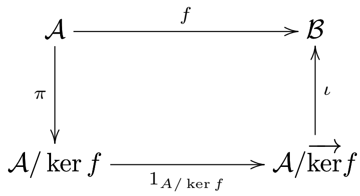
$$b = g(a \wedge a'),\ a \wedge a' \leq_A a',\ g(a') = b'.$$

□

6. The (E, M)-factorizations

With the knowledge that we have obtained we can now reformulate Theorem 1 in categorical terms.

Theorem 5. *For any homomorphism $f : \mathcal{A} \to \mathcal{B}$ in a variety $\mathcal{V}$ of ordered Ω-algebras, we have in $\mathcal{V}$ a commutative diagram*

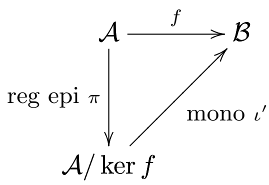


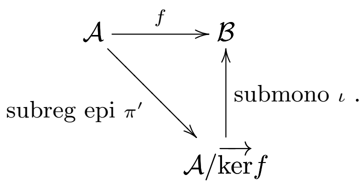
where $\pi : a \mapsto [a]$, $\iota : [a] \mapsto f(a)$. Moreover,

(1) *π is a regular epimorphism;*
(2) *$1_{A/\ker f}$ is a monomorphism;*
(3) *ι is a submonomorphism;*
(4) *$1_{A/\ker f} \circ \pi$ is a subregular epimorphism;*
(5) *$\iota \circ 1_{A/\ker f}$ is a monomorphism.*

If f is surjective then ι is an isomorphism.

Denoting $\pi' = 1_{A/\ker f} \circ \pi$ and $\iota' = \iota \circ 1_{A/\ker f}$ we have the following factorizations for each f in $\mathcal{V}$:





Let us denote by *RegEpi* (*SubRegEpi*) the class of all (sub)regular epimorphisms in a variety $\mathcal{V}$, and by *Mono* (*SubMono*) the class of all (sub)monomorphisms in $\mathcal{V}$. It turns out each variety has certain factorization systems.

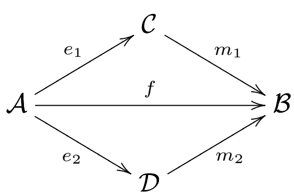
Theorem 6. *Every variety $\mathcal{V}$ of ordered Ω-algebras is*

(1) *(RegEpi, Mono)-structured;*
(2) *(SubRegEpi, SubMono)-structured.*

Proof. We use Proposition 14.7 of [1] to prove our theorem. In both cases conditions (1) and (2) of that proposition are obviously satisfied. By Theorem 5 we have the required factorizations, so we only need to prove that these factorizations are unique.

Consider first a commutative diagram

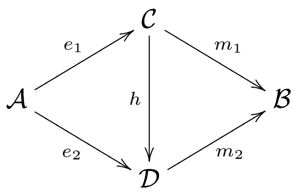
(6.1)



in $\mathcal{V}$, where e_1, e_2 are regular epimorphisms and m_1, m_2 are monomorphisms. We define a mapping $h : C \to D$ by

$$h(c) = e_2(a), \tag{6.2}$$

where $c \in C$ and $c = e_1(a)$ for some $a \in A$. It is easy to see that h is a well defined homomorphism of Ω-algebras and the diagram



commutes. From commutativity of (6.1) and injectivity of m_1, m_2 we deduce that $\ker e_1 = \ker e_2$. Suppose that $c \leq c'$ in C. Since e_1 is a regular epimorphism, there exist $a, a' \in A$ such that

$$c = e_1(a),\ a \underset{\ker e_1}{\leq} a',\ e_1(a') = c'. \tag{6.3}$$

Using the equalities $\ker e_1 = \ker e_2$ and $h \circ e_1 = e_2$ we conclude from (6.3) that $h(c) \leq h(c')$. This shows that h is monotone. In a similar manner we can construct a morphism $h' : D \to C$ such that $h' \circ h = 1_C$ and $h \circ h' = 1_D$. Thus h is an isomorphism.

For the second claim let us consider the diagram (6.1) where e_1, e_2 are surjective epimorphisms and m_1, m_2 are order embeddings. We define $h : C \to D$ again by (6.2). The only difference in the proof is in showing that h is monotone. If $c \leq c'$ in C, where $c = e_1(a)$ and $c' = e_1(a')$ for some $a, a' \in A$, then $m_1(c) \leq m_1(c')$, so $f(a) \leq f(a')$ or $m_2(e_2(a)) \leq m_2(e_2(a'))$. Since m_2 is an order embedding, we conclude that $e_2(a) \leq e_2(a')$, i.e. $h(c) \leq h(c')$. □

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Institute of Mathematics, Faculty of Mathematics and Computer Science, J. Liivi 2, University of Tartu, 50409 Tartu, Estonia
E-mail address: vlaan@ut.ee

Current address: Institute of Mathematics, Faculty of Mathematics and Computer Science, J. Liivi 2, University of Tartu, 50409 Tartu, Estonia, Author Two current address line 2
E-mail address: snasir@ut.ee