

THE MONOID OF HOLOMORPHIC ENDOMORPHISMS OF A GROUP AND ITS AUTOMORPHISMS

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This paper is dedicated to Ulrich Knauer and Mati Kilp

ABSTRACT. The holomorph of a group is a well-known concept in group theory. We provide here a generalization of the concept, which we call the monoid of holomorphic endomorphisms of a group. We characterize the automorphisms of this monoid for a general group, and we effectively calculate the automorphisms of the monoid of holomorphic endomorphisms of the group of integers. The critical tool to prove these results is a theoretical description of the automorphisms of the monoid of endomorphisms of transformations on a set X that preserve a reflexive relation on X .

1. INTRODUCTION

For a semigroup S , denote by $\text{Aut}(S)$ the group of automorphisms of S . A significant amount of research has been devoted to the study of $\text{Aut}(S)$ in the case when $S = \text{End}(\mathcal{A})$ is the monoid of endomorphisms of a mathematical structure $\mathcal{A}$. For example, Schreier [25] and Mal'cev [21] described all automorphisms of $\text{End}(X)$, where X is a set, and Gluskin [10, 11] described the automorphisms of, respectively, $\text{End}(X, \leq)$ and $\text{End}(V)$, where $(X, \leq)$ is a poset and V is a vector space. More examples are provided, among others, by Formanek [8], Levi [15, 16], Liber [19], Magill [20], Mashevitzky and Schein [22], Schein [24], Sullivan [27], and Šutov [28]. The authors have contributed to this line of research by studying the automorphism groups of $\text{End}(\mathcal{A})$ for: $\mathcal{A} = (X, \rho, R)$, where X is a set, ρ is an equivalence relation on X , and R is a cross-section of X/ρ [3]; $\mathcal{A} = B_{\mathcal{V}}(X)$, where X is a set, $\mathcal{V}$ is a variety of bands, and $B_{\mathcal{V}}(X)$ is the free object in $\mathcal{V}$ on X [4]; $\mathcal{A} = \mathcal{A}_{\mathcal{V}}(X)$, where X is a set, $\mathcal{V}$ is a variety of universal algebras, and $\mathcal{A}_{\mathcal{V}}(X)$ is the 1-simple free algebra in $\mathcal{V}$ on X [6]; $\mathcal{A} = D$, where D is a reflexive digraph [2]; and $\mathcal{A} = (X, \rho)$, where ρ is a reflexive n -ary relation on a set X [5]. In [1], a general algorithm for finding all automorphisms of a finite semigroup is provided. This algorithm has been implemented in the GAP [9] package MONOID [23].

Let ρ be a reflexive n -ary relation on a set X . In [5], the authors provided a theoretical description of the automorphisms of the monoid $T(X, \rho)$ consisting of all transformations of X that preserve ρ . This result can be used to describe the automorphisms of $\text{End}(\mathcal{A})$, where $\mathcal{A}$ is a mathematical structure, provided the endomorphisms of $\mathcal{A}$ can be interpreted as the elements of $T(X, \rho)$ where X is an underlying set of $\mathcal{A}$ and ρ is some reflexive n -ary relation on X .

In this paper, we introduce the monoid $\text{Hol}_E(G)$ of holomorphic endomorphisms of a group G , which is a generalization of the holomorph of G . We then define a

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reflexive quaternary relation ρ on G and prove that $Hol_E(G)$ is equal to $T(G, \rho)$. With the results presented in [5], this enables us to describe the automorphisms of $Hol_E(G)$. Using this general result, we prove that the group of automorphisms of $Hol_E(\mathbb{Z})$ is isomorphic to the holomorph of $\mathbb{Z}$ (where $\mathbb{Z}$ is the group of integers). We conclude the paper with some problems.

We assume that the reader is familiar with basic results of semigroup and group theories. As references we suggest [7, 13].

2. PRELIMINARIES

Let S be a semigroup. Any bijection $\phi : S \rightarrow S$ such that $(ab)\phi = (a\phi)(b\phi)$ for all $a, b \in S$ is called an *automorphism* of S . (We write mappings on the right and compose from left to right, that is, $a\phi$, rather than $\phi(a)$, and $a(\phi\psi) = (a\phi)\psi$, rather than $(\psi\phi)(a) = \psi(\phi(a))$.) The group of automorphisms of S will be denoted by $Aut(S)$.

For a set X , denote by $T(X)$ the monoid of full transformations on X , which consists of all mappings from X to X , with function composition as multiplication. The monoid $T(X)$ has the symmetric group $S(X)$ of all permutations on X as its group of units.

Given a subsemigroup S of $T(X)$, we denote by $\Delta(S)$ the subgroup of the symmetric group $S(X)$ consisting of all permutations g on X such that $g^{-1}Sg = S$. Subgroups G of $S(X)$ such that $G = \Delta(S)$ for some transformation semigroup S have been studied by Levi [17, 18]. Note that every element $g \in \Delta(S)$ induces an automorphism ϕ^g of S defined by $a\phi^g = g^{-1}ag$ for every $a \in S$.

Let ρ be a n -ary relation on a set X . We define $T(X, \rho)$ to be the set of all $a \in T(X)$ that preserve ρ , that is,

$$T(X, \rho) = \{a \in T(X) : (x_1, \dots, x_n) \in \rho \Rightarrow (x_1a, \dots, x_na) \in \rho\}.$$

It is clear that $T(X, \rho)$ is a submonoid of $T(X)$. When ρ is the universal n -ary relation on X , that is, ρ consists of all n -tuples of elements of X , then $T(X, \rho) = T(X)$. We also have $T(X, \rho) = T(X)$ when ρ is the identity relation on X . An n -ary relation ρ on X is said to be *reflexive* if $(x, x, \dots, x) \in \rho$ for every $x \in X$.

The following two results will be the main tool used in the next section.

Theorem 2.1. ([5, Theorem 2.1]) *Let ρ be a reflexive n -ary relation on X . Then*

$$Aut(T(X, \rho)) = \{\phi^g : g \in \Delta(T(X, \rho))\}.$$

Theorem 2.2. ([5, Theorem 2.2]) *Let ρ be a reflexive n -ary relation on X . Then $Aut(T(X, \rho))$ is isomorphic to $\Delta(T(X, \rho))$.*

3. HOLOMORPHIC ENDOMORPHISMS OF A GROUP

This section has two main goals. First, we introduce the concept of holomorphic endomorphisms of a group, thus providing a generalization of a very well-known concept of group theory. Second, we provide a theoretical description of the automorphisms of the monoid of holomorphic endomorphisms of a group.

Let G be a group. For $g \in G$, the right translation defined by g is the permutation $r_g \in S(G)$ defined by $xr_g = xg$ ($x \in G$). The set $\overline{G} = \{r_g : g \in G\}$ is a subgroup of $S(G)$ isomorphic to G , called the right regular representation of G .

The *holomorph* of a group G , denoted $Hol(G)$, is the normalizer of $\overline{G}$ in $S(G)$ [7, 12, 14], that is,

$$Hol(G) = N_{S(G)}(\overline{G}) = \{\sigma \in S(G) : \sigma^{-1}\overline{G}\sigma = \overline{G}\}.$$

Alternatively, $Hol(G)$ can be defined as the product of $Aut(G)$ and $\overline{G}$ [14], that is,

$$Hol(G) = Aut(G)\overline{G} = \{\phi r_g : \phi \in Aut(G), g \in G\},$$

or as the semidirect product of $Aut(G)$ and G , where $Aut(G)$ acts on G in the obvious way [26].

Since $Hol(G) = Aut(G)\overline{G}$, it is natural to extend the group $Hol(G)$ to the monoid

$$End(G)\overline{G} = \{\alpha r_g : \alpha \in End(G), g \in G\}.$$

(The set $End(G)\overline{G}$ is a submonoid of the full transformation monoid $T(G)$ since for all $\alpha, \beta \in End(G)$ and $g, h \in G$, $(\alpha r_g)(\beta r_h) = (\alpha\beta)r_{(g\beta)h}$.) We shall denote the monoid $End(G)\overline{G}$ by $Hol_E(G)$ and call its elements *holomorphic endomorphisms* of G . Note that $Hol(G)$ is the group of units of $Hol_E(G)$. Also note that $Aut(G) \subseteq Hol(G)$ and $End(G) \subseteq Hol_E(G)$.

Our objective is to use Theorem 2.1 to prove that every automorphism of the monoid $Hol_E(G)$ is induced by a permutation $\sigma \in S(G)$ such that $\sigma^{-1}Hol_E(G)\sigma = Hol_E(G)$. To this end, we fix the following quaternary relation ρ on G :

$$(3.1) \quad \rho = \{(x, y, z, w) : w = xy^{-1}z\}.$$

It is obvious that ρ is a reflexive relation on G . The next result provides a characterization of the holomorphic endomorphisms and also links the problem of describing $Aut(Hol_E(G))$ to Theorems 2.1 and 2.2.

Proposition 3.1. *Let G be a group. Then $Hol_E(G) = T(G, \rho)$.*

Proof. Recall that $Hol_E(G) = End(G)\overline{G}$. To prove that $End(G)\overline{G} \subseteq T(G, \rho)$, let $(x, y, z, w) \in \rho$ and $\alpha r_g \in End(G)\overline{G}$ ($\alpha \in End(G)$, $g \in G$). Then we have:

$$\begin{aligned} w(\alpha r_g) &= (xy^{-1}z)(\alpha r_g) \\ &= ((x\alpha)(y^{-1}\alpha)(z\alpha))r_g \\ &= (x\alpha)gg^{-1}(y\alpha)^{-1}(z\alpha)g \\ &= (x\alpha)g((y\alpha)g)^{-1}(z\alpha)g \\ &= (x(\alpha r_g))(y(\alpha r_g))^{-1}(z(\alpha r_g)), \end{aligned}$$

so $(x(\alpha r_g), y(\alpha r_g), z(\alpha r_g), w(\alpha r_g)) \in \rho$. Thus $Hol_E(G) = End(G)\overline{G} \subseteq T(G, \rho)$.

For the reverse inclusion, let $f \in T(G, \rho)$. We want to show that $f = \alpha r_g$ for some $\alpha \in End(G)$ and $g \in G$. Let $g = 1f$ (where 1 is the identity of G) and let $\alpha : G \rightarrow G$ be defined by $x\alpha = (xf)g^{-1}$. Then clearly $f = \alpha r_g$. It remains to show that α is an endomorphism of G . Let $x, y \in G$. Since $f \in T(G, \rho)$ and $(x, 1, y, xy) \in \rho$, we have $(xf)g^{-1}(yf) = (xf)(1f)^{-1}(yf) = (xy)f$, so $(xf)g^{-1}(yf)g^{-1} = ((xy)f)g^{-1}$. Hence

$$(x\alpha)(y\alpha) = (xf)g^{-1}(yf)g^{-1} = ((xy)f)g^{-1} = (xy)\alpha.$$

Thus $\alpha \in End(G)$, and so $T(G, \rho) \subseteq End(G)\overline{G} = Hol_E(G)$. $\square$

As $Hol(G)$ is the group of units of $Hol_E(G)$, Proposition 3.1 immediately yields a new characterization of the holomorph of a group.

Corollary 3.2. *Let G be a group and ρ be the quaternary relation on G defined by (3.1). Then*

$$\text{Hol}(G) = \{\sigma \in S(G) : \forall x, y, z, w \in X((x, y, z, w) \in \rho \Rightarrow (x\sigma, y\sigma, z\sigma, w\sigma) \in \rho)\}.$$

Recall that $\Delta(\text{Hol}_E(G))$ is the group of all permutations $\sigma \in S(G)$ that preserve $\text{Hol}_E(G)$ under conjugation, that is,

$$\Delta(\text{Hol}_E(G)) = \{\sigma \in S(G) : \sigma^{-1}\text{Hol}_E(G)\sigma = \text{Hol}_E(G)\},$$

and that for $\sigma \in \Delta(\text{Hol}_E(G))$, ϕ^σ denotes the inner automorphism of $\text{Hol}_E(G)$ induced by σ , that is, $\gamma\phi^\sigma = \sigma^{-1}\gamma\sigma$ (where $\gamma \in \text{Hol}_E(G)$). The following theorem asserts that all automorphisms of the monoid $\text{Hol}_E(G)$ of holomorphic endomorphisms of G are inner and that the group of automorphisms of $\text{Hol}_E(G)$ is isomorphic to the group $\Delta(\text{Hol}_E(G))$.

Theorem 3.3. *Let G be a group. Then:*

- (1) $\text{Aut}(\text{Hol}_E(G)) = \{\phi^\sigma : \sigma \in S(G) \text{ and } \sigma^{-1}\text{Hol}_E(G)\sigma = \text{Hol}_E(G)\}.$
- (2) $\text{Aut}(\text{Hol}_E(G)) \cong \Delta(\text{Hol}_E(G)).$

Proof. It follows immediately from Proposition 3.1, Theorem 2.1, and Theorem 2.2. $\square$

Since $\text{Aut}(\text{Hol}_E(G)) \cong \Delta(\text{Hol}_E(G))$, it would be of interest to describe $\Delta(\text{Hol}_E(G))$ for various classes of groups G , for example cyclic, finite, abelian, or symmetric. Since $\text{Hol}(G)$ is the group of units of the monoid $\text{Hol}_E(G)$, it is clear that $\text{Hol}(G)$ is a subgroup of $\Delta(\text{Hol}_E(G))$. Therefore, it would also be of interest to characterize the groups G for which $\Delta(\text{Hol}_E(G)) = \text{Hol}(G)$. Next section is devoted to the proof that this holds for the group of integers.

4. THE AUTOMORPHISMS OF THE HOLOMORPHIC ENDOMORPHISMS OF THE GROUP OF INTEGERS

The aim of this section is to prove that the group $\mathbb{Z}$ of integers has the following property:

$$\Delta(\text{Hol}_E(\mathbb{Z})) = \text{Hol}(\mathbb{Z}),$$

and hence

$$\text{Aut}(\text{Hol}_E(\mathbb{Z})) \cong \text{Hol}(\mathbb{Z}).$$

For $k \in \mathbb{Z}$, define $f_k : \mathbb{Z} \rightarrow \mathbb{Z}$ by $xf_k = kx$. It is well known that $\text{Aut}(\mathbb{Z}) = \{f_1, f_{-1}\}$ and $\text{End}(\mathbb{Z}) = \{f_k : k \in \mathbb{Z}\}$. Therefore $\text{Hol}_E(\mathbb{Z}) = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$ is the monoid of all mappings $f_{k,z} : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by $xf_{k,z} = kx + z$, where $k, z \in \mathbb{Z}$. (That is, $\text{Hol}_E(\mathbb{Z})$ consists of all linear polynomial functions from $\mathbb{Z}$ to $\mathbb{Z}$.)

Lemma 4.1. *Let $x \in \mathbb{Z}$ and suppose that $\sigma \in S(\mathbb{Z})$ is such that $\sigma^{-1}r_x\sigma \in f_e\overline{\mathbb{Z}}$, where $e \in \{1, -1\}$. Then there exists $z \in \mathbb{Z}$ such that $(i+x)\sigma = e(i\sigma) + z$ for all $i \in \mathbb{Z}$.*

Proof. Since $\sigma^{-1}r_x\sigma \in f_e\overline{\mathbb{Z}}$, there is $z \in \mathbb{Z}$ such that $\sigma^{-1}r_x\sigma = f_e r_z$. Thus for all $i \in \mathbb{Z}$, $i(r_x\sigma) = (i\sigma)(\sigma^{-1}r_x\sigma) = (i\sigma)(f_e r_z)$, and so $(i+x)\sigma = i(r_x\sigma) = (i\sigma)(f_e r_z) = e(i\sigma) + z$. $\square$

In what follows, Lemma 4.1 will be used frequently without any further reference.

Lemma 4.2. *Let $\sigma \in S(\mathbb{Z})$ be such that $\sigma^{-1}\text{End}(\mathbb{Z})\overline{\mathbb{Z}}\sigma = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$. Then $\sigma^{-1}r_x\sigma \in \text{Aut}(\mathbb{Z})\overline{\mathbb{Z}}$ for every $x \in \mathbb{Z}$.*

Proof. Let $x \in \mathbb{Z}$. Then $\sigma^{-1}r_x\sigma = \alpha r_z$ for some $\alpha \in \text{End}(\mathbb{Z})$ and some $z \in \mathbb{Z}$. Thus, since $\alpha = \sigma^{-1}r_x\sigma r_z^{-1} \in S(\mathbb{Z})$, we have $\alpha \in \text{End}(\mathbb{Z}) \cap S(\mathbb{Z}) = \text{Aut}(\mathbb{Z})$, and so $\sigma^{-1}r_x\sigma = \alpha r_z \in \text{Aut}(\mathbb{Z})\overline{\mathbb{Z}}$. $\square$

Lemma 4.3. *Let $\sigma \in S(\mathbb{Z})$ be such that $\sigma^{-1}\text{End}(\mathbb{Z})\overline{\mathbb{Z}}\sigma = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$ and let $x \in \mathbb{Z}$. Suppose that $\sigma^{-1}r_x\sigma \in f_{-1}\overline{\mathbb{Z}}$. Then $\sigma^{-1}r_y\sigma \in f_{-1}\overline{\mathbb{Z}}$ for every $y \in \mathbb{Z} - \{0\}$.*

Proof. Let $\sigma^{-1}r_x\sigma = f_{-1}r_z$ ($z \in \mathbb{Z}$) and let $y \in \mathbb{Z}$. By Lemma 4.2, $\sigma^{-1}r_y\sigma \in \text{Aut}(\mathbb{Z})\overline{\mathbb{Z}}$. Suppose that $\sigma^{-1}r_y\sigma = f_1r_w$ ($w \in \mathbb{Z}$). We claim that $y = 0$. Indeed, $\sigma^{-1}r_x\sigma = f_{-1}r_z$ implies $(i+x)\sigma = -(i\sigma) + z$ for every $i \in \mathbb{Z}$, and hence $z = x\sigma + 0\sigma$ (take $i = 0$). Similarly, $\sigma^{-1}r_y\sigma = f_1r_w$ implies $(i+y)\sigma = (i\sigma) + w$ for every $i \in \mathbb{Z}$, and hence $w = y\sigma - 0\sigma$. Thus $(y+x)\sigma = -(y\sigma) + z = -(y\sigma) + x\sigma + 0\sigma$. On the other hand, $(y+x)\sigma = (x+y)\sigma = x\sigma + w = x\sigma + y\sigma - 0\sigma$. Therefore, $-(y\sigma) + x\sigma + 0\sigma = x\sigma + y\sigma - 0\sigma$, and hence $2(0\sigma) = 2(y\sigma)$. Hence $y = 0$ (since σ is one-to-one), and the lemma follows. $\square$

Lemma 4.4. *Let $\sigma \in S(\mathbb{Z})$ be such that $\sigma^{-1}\text{End}(\mathbb{Z})\overline{\mathbb{Z}}\sigma = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$ and let $x, y \in \mathbb{Z}$. Suppose that $\sigma^{-1}r_x\sigma \in \overline{\mathbb{Z}}$. Then $(x+y)\sigma = x\sigma + y\sigma - 0\sigma$.*

Proof. Suppose that $x \neq 0$. By Lemma 4.2, $\sigma^{-1}r_y\sigma \in f_{-1}\overline{\mathbb{Z}}$ or $\sigma^{-1}r_y\sigma \in f_1\overline{\mathbb{Z}} = \overline{\mathbb{Z}}$. But since $x \neq 0$, the former cannot happen by Lemma 4.3. Thus $\sigma^{-1}r_y\sigma = r_w$ for some $w \in \mathbb{Z}$. Hence $(i+y)\sigma = i\sigma + w$ for every $i \in \mathbb{Z}$, and so $w = y\sigma - 0\sigma$. Therefore, if $x \neq 0$, we have $(x+y)\sigma = x\sigma + w = x\sigma + y\sigma - 0\sigma$. If $x = 0$ then $(0+y)\sigma = y\sigma = 0\sigma + y\sigma - 0\sigma$, and so the result is true in all cases. $\square$

Lemma 4.5. *Let $\sigma \in S(\mathbb{Z})$ be such that $\sigma^{-1}\text{End}(\mathbb{Z})\overline{\mathbb{Z}}\sigma = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$ and let $x \in \mathbb{Z} - \{0\}$. Then $\sigma^{-1}r_x\sigma \in \overline{\mathbb{Z}}$.*

Proof. Suppose that $\sigma^{-1}r_x\sigma \in f_{-1}\overline{\mathbb{Z}}$. Let $y \in \mathbb{Z} - \{x, 0\}$. By Lemma 4.3, $\sigma^{-1}r_y\sigma \in f_{-1}\overline{\mathbb{Z}}$. Thus $\sigma^{-1}r_x\sigma = f_{-1}r_z$ and $\sigma^{-1}r_y\sigma = f_{-1}r_w$ for some $z, w \in \mathbb{Z}$. As in the proof of Lemma 4.3, we have $z = x\sigma + 0\sigma$ and $w = y\sigma + 0\sigma$. Thus $(x+y)\sigma = -(x\sigma) + w = -(x\sigma) + y\sigma + 0\sigma$ and $(y+x)\sigma = -(y\sigma) + z = -(y\sigma) + x\sigma + 0\sigma$. Since $(x+y)\sigma = (y+x)\sigma$, we have $-(x\sigma) + y\sigma + 0\sigma = -(y\sigma) + x\sigma + 0\sigma$, and hence $2(y\sigma) = 2(x\sigma)$, so $y = x$. This is a contradiction that follows from the assumption that we could have $\sigma^{-1}r_x\sigma \in f_{-1}\overline{\mathbb{Z}}$. Thus we must have $\sigma^{-1}r_x\sigma \in \overline{\mathbb{Z}}$ by Lemma 4.2. $\square$

Theorem 4.6. *Let $\mathbb{Z}$ be the group of integers. Then:*

- (1) $\Delta(\text{Hol}_E(\mathbb{Z})) = \text{Hol}(\mathbb{Z})$.
- (2) $\text{Aut}(\text{Hol}_E(\mathbb{Z})) \cong \text{Hol}(\mathbb{Z})$.

Proof. We already observed that $\text{Hol}(G) \subseteq \Delta(\text{Hol}_E(G))$ for any group G . (See the paragraph after the proof of Theorem 3.3.) For the reverse inclusion, let $\sigma \in \Delta(\mathbb{Z})$, that is, $\sigma \in S(\mathbb{Z})$ and $\sigma^{-1}\text{End}(\mathbb{Z})\overline{\mathbb{Z}}\sigma = \text{End}(\mathbb{Z})\overline{\mathbb{Z}}$. Define $\gamma : \mathbb{Z} \rightarrow \mathbb{Z}$ by $x\gamma = x\sigma - 0\sigma$. It is clear that $\gamma \in S(\mathbb{Z})$ and that $0\gamma = 0$. Let $x, y \in \mathbb{Z}$. If $x = 0$ then $(x+y)\gamma = (0+y)\gamma = y\gamma = 0 + y\gamma = 0\gamma + y\gamma = x\gamma + y\gamma$. If $x \neq 0$ then $(x+y)\sigma = x\sigma + y\sigma - 0\sigma$ (by Lemma 4.5 and Lemma 4.4), and so $(x+y)\gamma = (x+y)\sigma - 0\sigma = x\sigma + y\sigma - 0\sigma - 0\sigma = x\sigma - 0\sigma + y\sigma - 0\sigma = x\gamma + y\gamma$. Thus $\gamma \in \text{Aut}(\mathbb{Z})$, and so $\sigma = \gamma r_{0\sigma} \in \text{Aut}(\mathbb{Z})\overline{\mathbb{Z}} = \text{Hol}(\mathbb{Z})$. This proves (1). Statement (2) follows immediately from (1) and Theorem 3.3. $\square$

5. PROBLEMS

1. The concept of holomorphic endomorphisms raises many interesting problems, for example: describe the structure, congruences, and automorphisms of the monoid of holomorphic endomorphisms of a finite (symmetric, alternating, free, abelian, etc.) group.
2. Let S be a regular semigroup. For every $s \in S$ fix an element $s^{-1} \in S$ such that $ss^{-1}s = s$. Then the relation

$$(5.1) \quad \rho = \{(x, y, z, w) \in S^4 : xy^{-1}z = w\}$$

is reflexive and hence we can apply Theorem 2.1 to $\text{Aut}(T(S, \rho))$. Describe the group $\text{Aut}(T(S, \rho))$ when S belongs to some class of regular semigroups such as bands, completely 0-simple semigroups, inverse semigroups, completely regular semigroups, etc.

3. We proved in Section 3 that if G is a group and ρ is the quaternary relation on G defined by (3.1), then $T(G, \rho) = \text{End}(G)\overline{G}$. Let S be a regular (or inverse) semigroup. Describe the relation between $T(S, \rho)$ and $\text{End}(S)\overline{S}$, where ρ is defined by (5.1) and $\overline{S}$ is the image of S under the right regular representation of S .

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REFERENCES

- [1] J. Araújo, P.V. Bunau, J.D. Mitchell, *Computing automorphisms of semigroups* (to appear).
- [2] J. Araújo, E. Dobson, J. Konieczny, *Automorphisms of endomorphism semigroups of reflexive digraphs* (to appear in Math. Nachr.).
- [3] J. Araújo, J. Konieczny, *Automorphism groups of centralizers of idempotents*, J. Algebra **269** (2003), 227–239.
- [4] J. Araújo, J. Konieczny, *Automorphisms of endomorphism monoids of relatively free bands*, Proc. Edinburgh Math. Soc. (2) **50** (2007), 1–21.
- [5] J. Araújo, J. Konieczny, *A method of finding automorphism groups of endomorphism monoids of relational systems*, Discrete Mathematics **307** (2007), 1609–1620.
- [6] J. Araújo, J. Konieczny, *Automorphisms of endomorphism monoids of 1-simple free algebras* (to appear in Comm. Algebra).
- [7] J.D. Dixon, B. Mortimer, *Permutation Groups*, Graduate Texts in Mathematics **163**, Springer-Verlag, New York, 1996.
- [8] E. Formanek, *A question of B. Plotkin about the semigroup of endomorphisms of a free group*, Proc. Amer. Math. Soc. **130** (2002), 935–937.
- [9] *The GAP Group* (GAP – Groups, Algorithms, and Programming), <http://www.gap-system.org>
- [10] L.M. Gluskin, *Semi-groups of isotone transformations* (Russian), Uspehi Mat. Nauk **16** (1961), 157–162.
- [11] L.M. Gluskin, *Semigroups and rings of endomorphisms of linear spaces I*, Amer. Math. Soc. Transl. **45** (1965), 105–137.
- [12] M. Hall, Jr., *The Theory of Groups*, The Macmillan Company, New York, 1959.
- [13] J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, New York, 1995.
- [14] A.G. Kurosh, *The Theory of Groups*, Volume One, Chelsea Publishing Company, New York, 1955.
- [15] I. Levi, *Automorphisms of normal transformation semigroups*, Proc. Edinburgh Math. Soc. (2) **28** (1985), 185–205.

- [16] I. Levi, *Automorphisms of normal partial transformation semigroups*, Glasgow Math. J. **29** (1987), 149–157.
- [17] I. Levi, *On the inner automorphisms of finite transformation semigroups*, Proc. Edinburgh Math. Soc. (2) **39** (1996), 27–30.
- [18] I. Levi, *On groups associated with transformation semigroups*, Semigroup Forum **59** (1999), 342–353.
- [19] A.E. Liber, *On symmetric generalized groups* (Russian), Mat. Sbornik N.S. **33** (1953), 531–544.
- [20] K.D. Magill, *Semigroup structures for families of functions, I. Some homomorphism theorems*, J. Austral. Math. Soc. **7** (1967), 81–94.
- [21] A.I. Mal'cev, *Symmetric groupoids* (Russian), Mat. Sbornik N.S. **31** (1952), 136–151.
- [22] G. Mashevitzky, B.M. Schein, *Automorphisms of the endomorphism semigroup of a free monoid or a free semigroup*, Proc. Amer. Math. Soc. **131** (2003), 1655–1660.
- [23] J.D. Mitchell, *The MONOID package v3.1*, <http://www-groups.mcs.st-andrews.ac.uk/~jamesm/semigroups/Monoid.html>.
- [24] B.M. Schein, *Ordered sets, semilattices, distributive lattices and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. **68** (1970), 31–50.
- [25] I. Schreier, *Über Abbildungen einer abstrakten Menge auf ihre Teilmengen*, Fund. Math. **28** (1936), 261–264.
- [26] W.R. Scott, *Group Theory*, Prentice-Hall, Englewood Cliffs, N.J., 1964.
- [27] R.P. Sullivan, *Automorphisms of transformation semigroups*, J. Austral. Math. Soc. **20** (1975), 77–84.
- [28] È.G. Šutov, *Homomorphisms of the semigroup of all partial transformations* (Russian), Izv. Vysš. Učebn. Zaved. Matematika **3** (1961), 177–184.

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