

FLATNESS PROPERTIES OF S -POSETS: AN OVERVIEW

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For Mati and Ulrich on the occasion of their 65th birthdays

ABSTRACT. For almost four decades, the study of monoid actions has been a flourishing area of research in algebra. The monograph *Monoids, Acts and Categories* by Kilp, Knauer and Mikhalev (Walter de Gruyter, Berlin, 2000) presents a detailed and comprehensive account of this research, and contains almost 500 bibliographic items, nearly 100 of which pertain to flatness properties of acts. There is also a considerable volume of literature that studies partially ordered sets from an algebraic viewpoint, investigating such notions as congruences and ideals, but to the author's knowledge, no up-to-date single work is yet available describing this research. In the mid 1980s these two streams were brought together with the appearance of two papers by Syed Fakhruddin on ordered monoids acting on posets (they were called S -posets, and the name has stuck). One article made a preliminary study of flatness of S -posets, and the other dealt with S -posets as a category. These works remained virtually unknown until around 2005, since which time a flurry of activity has ensued, and some dozen new research articles on flatness properties of S -posets have appeared. The present paper attempts to give an account of what is known so far about this subject.

1. INTRODUCTION AND PRELIMINARIES

1.1. Introduction. A *pomonoid* is a monoid S equipped with a partial order relation $\leq$ that is compatible in the sense that $s \leq t$ and $u \leq v$ imply $su \leq tv$, for all $s, t, u, v \in S$. A *right S -act* over a monoid S , often denoted A_S , is a non-empty set A equipped with a right S -action $(a, s) \rightsquigarrow as$ (for $a \in A, s \in S$) such that $a(st) = (as)t$ and $a1 = a$ for all $a \in A$ and all $s, t \in S$. If A_S is a right S -act where S is a pomonoid, and if A is also a poset such that the S -action is monotone in both variables, then A_S is called a *right S -poset*. Left S -acts and left S -posets are defined similarly, and categories denoted $Act-S$, $S-Act$, (resp. $Pos-S$ and $S-Pos$) result, in which the morphisms are the obvious action-preserving (resp. action- and order-preserving) maps. A comprehensive treatment of S -acts is the subject of [16]. The purpose of the present article is to describe progress that has been made so far in studying S -posets. The reader should consult [15], for example, for general results on semigroup theory; [1] will be our standard reference for category-theoretic notions.

Just as S -acts provide another way of looking at representations of monoids by full transformation monoids of sets, S -posets correspond exactly to order-preserving representations of pomonoids by monoids of monotone transformations of posets. In more detail, suppose A is a poset. Denote by $\mathcal{O}(A)$ the pomonoid (under left

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to right composition and pointwise order) of all monotone self maps of A . For a pomonoid S , if $\varphi : S \rightarrow \mathcal{O}(A)$ is a pomonoid homomorphism, then A becomes a right S -poset when equipped with the action $a \cdot s = a\varphi(s)$, and every right S -poset is isomorphic to one obtained in this fashion.

1.2. S -poset congruences. If A is an algebraic structure and θ an equivalence relation on A , one attempts to impose extra conditions on θ so that the quotient set A/θ becomes an algebraic structure of the same “type” as A in such a way that the natural map $A \rightarrow A/\theta$ is a homomorphism. Such constructions are extremely well known in all branches of algebra, and correspond to normal subgroups in group theory, ideals in ring theory, vector subspaces in linear algebra, and so forth, and the appropriate equivalence relations are called *congruences*. We now consider the notion of congruence for S -posets.

First, let A be a poset. For a reflexive relation ρ on A , we define the quasiorder $\leq_\rho$ on A by

$$a \leq_\rho b \text{ if there exist } a_i, b_i \in A \text{ such that } a \leq a_1 \rho b_1 \leq a_2 \rho b_2 \leq \cdots \leq a_n \rho b_n \leq b.$$

An equivalence relation θ on A is called an *order congruence* if a partial order on the quotient set A/θ exists such that the natural map $A \rightarrow A/\theta$ is monotone. In [8] and elsewhere, intrinsic conditions that θ be an order congruence are determined, as follows: an equivalence relation θ is an order congruence on A if and only if the implication

$$a \leq_\theta b \leq_\theta a \text{ implies } a\theta b$$

holds, for all $a, b \in A$. If θ is an order congruence on A , then the *standard order* on A/θ is given by

$$a\theta \leq b\theta \text{ if and only if } a \leq_\theta b,$$

and it is easily checked that the standard order is the coarsest order on A/θ such that the natural map is monotone. The analysis above also works if A_S is an S -poset on which θ is an S -act congruence. In this case, if θ is also an order congruence, it will be called an *S -poset congruence*. The S -act A/θ with its standard order will be an S -poset, the natural map $A \rightarrow A/\theta$ will be an S -poset homomorphism, and the standard order will again be the coarsest compatible order on A/θ .

Another analysis of S -poset congruences appears in [24], wherein the notion of pseudoorder considered for pomonoids in [17] (and appearing earlier under different names in [9], [10]) is adapted to S -posets. The *directed kernel* of an S -poset homomorphism $\varphi : A_S \rightarrow C_S$ is the relation

$$\overrightarrow{\ker \varphi} := \{(a, a') \in A \times A : \varphi(a) \leq \varphi(a')\}.$$

It is easily checked a directed kernel is a right S -compatible, transitive relation on A that contains the order relation $\leq$. A relation with these three properties is called a *pseudoorder* on A_S . That each pseudoorder on A_S is in fact a directed kernel can be seen as follows: if σ is a pseudoorder, then the relation $\theta = \sigma \cap \sigma^{-1}$ is an S -poset congruence on A_S , the relation on A/θ defined by

$$a\theta \leq b\theta \text{ if and only if } a\sigma b$$

makes A/θ into an S -poset, and the directed kernel of the natural map $A \rightarrow A/\theta$ is exactly σ . For a given S -poset congruence θ on A_S it is of course possible that

there are many pseudoorders σ on A such that $\theta = \sigma \cap \sigma^{-1}$ (and the standard order on A/θ given by

$$a\theta \leq b\theta \text{ if and only if } a \leq_{\frac{\theta}{\theta}} b$$

may well be strictly coarser than the order just defined). The following example illustrates this possibility.

Example 1.1. (1) Let S be a 1-element monoid and let A_S be the 3-element S -poset $\{a, b, c\}$ with the discrete partial order. Let θ be the S -act congruence $\Delta \cup \{(a, b), (b, a)\}$ on A_S , where Δ denotes the equality relation on A . We may consider the following S -poset morphisms $\varphi_i : A_S \rightarrow C_S$ and their associated directed kernels σ_i ($i = 1, 2, 3$), for different choices of C_S :

- (i) C_S is a discrete two-element set $\{x, y\}$, $\varphi_1(a) = \varphi_1(b) = x$, $\varphi_1(c) = y$, and

$$\sigma_1 = \theta.$$

- (ii) C_S is a two-element chain $0 < 1$, $\varphi_2(a) = \varphi_2(b) = 0$, $\varphi_2(c) = 1$, and

$$\sigma_2 = \theta \cup \{(a, c), (b, c)\}.$$

- (iii) C_S is a two-element chain $0 < 1$, $\varphi_3(a) = \varphi_3(b) = 1$, $\varphi_3(c) = 0$, and

$$\sigma_3 = \theta \cup \{(c, a), (c, b)\}.$$

Then, in each case, θ is an S -poset congruence, with $\theta = \sigma_i \cap \sigma_i^{-1}$ and the standard order on A/θ is the discrete order.

(2) For θ to be an S -poset congruence, it is necessary that the θ -classes be convex: if $a \leq b \leq c$ and $a\theta c$, then we have both $a \leq_{\frac{\theta}{\theta}} b$ and $b \leq_{\frac{\theta}{\theta}} a$, so $a\theta b$ results. The convexity condition is not however sufficient, for suppose A is the poset $\{a, b, c, d\}$ with $a < c$, $b < d$, $a\theta d$, and $b\theta c$ being the only relations. Then $a \leq_{\frac{\theta}{\theta}} b$ and $b \leq_{\frac{\theta}{\theta}} a$, but a and b are not θ -related.

In [4], and later in [22], if A_S is an S -poset and $H \subseteq A \times A$, a description is given of the S -poset congruence $\nu(H)$ on A_S induced by H . If π denotes the natural S -poset homomorphism $A \rightarrow A/\nu(H)$, then $\nu(H)$ is characterized by the two properties

- (i) $H \subseteq \overrightarrow{\ker \pi}$, and

(ii) if $H \subseteq \overrightarrow{\ker \varphi}$ for some S -poset morphism with domain A_S , then $\nu(H) \subseteq \ker \varphi$. So, in a sense, $\nu(H)$ is the least S -poset congruence on A_S that “respects” the relations imposed by H . A construction of $\nu(H)$ is provided in [4], but it is perhaps instructive to note that if one takes σ to be the transitive closure of the relation $(\leq) \cup HS$ on A , then σ is precisely the relation $\leq_{\alpha(H)}$ of [4] and $\nu(H)$ is exactly

$\sigma \cap \sigma^{-1}$. Finally, the smallest S -poset congruence on A_S identifying all elements of H is $\theta(H) = \nu(H \cup H^{-1})$. The relation $\theta(H)$ is called the S -poset congruence on A_S generated by H . If H is an equivalence relation on A , then $\theta(H) = \nu(H)$.

1.3. Category-theoretic matters. A study of S -posets from a category-theoretic standpoint forms the content of [5], and extends (and in some cases, corrects) the results found in [14]. In the present survey, we shall only touch on some of the main points. Taking [1] as our guide to category theory, we find that in the category $Pos\text{-}S$, where S is a pomonoid, there are basically two kinds of monomorphisms and two kinds of epimorphisms.

Ordinary monomorphisms are simply the one-to-one S -poset maps. On the other hand, the monomorphisms in $Pos\text{-}S$ that are variously labelled *regular* (that is, equalizers), *extremal* (meaning any right factor that is an epimorphism is actually an isomorphism), *S -poset embeddings over Set* , or *S -poset embeddings over $Act\text{-}S$* all turn out to be the same: they are the order-embeddings (which reflect as well as preserve order).

For epimorphisms, the ordinary ones are exactly the surjective S -poset maps (contrary to what is stated in [14]), whereas the epimorphisms $g : A_S \rightarrow B_S$ that are called *regular*, *extremal* or *quotient maps* (over Set or $Act\text{-}S$) again all coincide and are characterized by the following property: whenever $b \leq b'$ in B , there exist $a_i, a'_i \in A$ such that

$$\begin{array}{ccccccc} b = g(a_1) & & g(a'_1) = g(a_2) & & & & g(a'_n) = b' \\ & a_1 \leq a'_1 & & a_2 \leq a'_2 & \cdots & a_n \leq a'_n & \end{array}.$$

In notation introduced earlier, this can be more succinctly stated as follows: an epimorphism $g : A_S \rightarrow B_S$ is regular if and only if

$$\overrightarrow{\ker g} = \underset{\ker g}{\leq}.$$

The simplest example of a non-regular epimorphism is obtained by taking $S = \{1\}$, A_S and B_S both to be the 2-element set $\{0, 1\}$ in which A carries the discrete order and B carries the usual order $0 < 1$, and taking g to be the identity map.

Products in $Pos\text{-}S$ are simply cartesian products with componentwise S -action and order, whereas coproducts are disjoint unions, with natural action and componentwise order. Equalizers, coequalizers, pullbacks and pushouts can all be constructed without difficulty (allowing empty objects when necessary).

In case A_S is a proper S -subposet of B_S the *amalgamated coproduct of two copies of B over A* , denoted $B \sqcup_A B$, is very useful in studying flatness, and has the following concrete realization. Let x, y and z be new symbols, and define

$$T = (\{x, y\} \times (B \setminus A)) \cup (\{z\} \times A).$$

Define a right S -action on T by

$$(w, c)s = \begin{cases} (z, cs) & \text{if } cs \in A, \\ (w, cs) & \text{otherwise.} \end{cases}$$

The partial order on T is given by

$$\begin{aligned} (w_1, c) &\leq (w_2, d) \text{ if and only if} \\ (w_1 = w_2 \text{ and } c \leq d) &\text{ or } (w_1 \neq w_2 \text{ and } c \leq a \leq d \text{ for some } a \in A). \end{aligned}$$

Then T_S is isomorphic to $B \sqcup_A B$. As a simple application of this construction, here is a proof that a non-surjective S -poset morphism $\varphi : A_S \rightarrow B_S$ is not an epimorphism. Simply let α and β denote the canonical mappings of B_S into $B_S \sqcup_{\varphi[A]} B_S$ and note that $\alpha \neq \beta$, even though $\alpha\varphi = \beta\varphi$.

In [5] some attention is paid to adjoint situations giving relations of the category $Pos\text{-}S$ with the categories Set , Pos , and $Act\text{-}S$. For the purposes of the present survey, we merely mention that by a *free S -poset on a poset P* we mean an S -poset F_S together with a monotone map $\epsilon : P \rightarrow F$ with the universal property that, given any S -poset A_S and any monotone map $\alpha : P \rightarrow A$ there exists a unique S -poset map $\beta : F \rightarrow A$ such that $\beta \circ \epsilon = \alpha$. The free S -poset on a poset P is uniquely determined up to isomorphism, and may be concretely represented as

$P \times S$, with action $(p, s)t = (p, st)$ and with componentwise order. In particular, if P is a discrete (unordered) set, then the free S -poset on P may be regarded as the coproduct of $|P|$ copies of the S -poset S_S .

1.4. Tensor products. Recall that the usual tensor product of S -acts is obtained as follows: if A_S and ${}_S B$ are given S -acts, one defines τ to be the smallest equivalence relation on the set $A \times B$ containing all pairs $((as, b), (a, sb))$ for $a \in A$, $b \in B$, and $s \in S$. The quotient set $(A \times B)/\tau$ is denoted $A \otimes B$ and is called the *tensor product of A_S and ${}_S B$* . The canonical image in $A \otimes B$ of a pair $(a, b) \in A \times B$ is denoted $a \otimes b$. The map $\otimes$ is *balanced* in the sense that $as \otimes b = a \otimes sb$ for all a, s , and b , and furthermore is universal among all balanced maps from $A \times B$ into sets. The tensor product of S -acts is uniquely determined up to (set) isomorphism.

A parallel construction for the tensor product of S -posets exists and can be performed using the ingredients we have introduced above. If A_S and ${}_S B$ are S -posets, we first consider $A \times B$ as a poset with the product order. Next, form the relation τ on $A \times B$ as described in the previous paragraph. Finally, construct the S -poset congruence $\theta(\tau) = \nu(\tau)$ on $A \times B$ generated by τ where the S -action on $A \times B$ is taken to be the trivial action. The resulting poset $(A \times B)/\theta(\tau)$ is the desired tensor product. The same notation $A \otimes B$ is used as for S -act tensor products, and similar remarks hold here concerning uniqueness and universal properties. In working with flatness properties of S -posets, the following description of the order relation in $A \otimes B$ is of crucial importance. (It first appeared explicitly in [23].)

Proposition 1.2. *Let A_S and ${}_S B$ be S -posets and suppose $a, a' \in A$ and $b, b' \in B$. Then $a \otimes b \leq a' \otimes b'$ in $A \otimes B$ if and only if*

$$\begin{array}{ccc} a & \leq & a_1 s_1 \\ a_1 t_1 & \leq & a_2 s_2 \quad \quad s_1 b \leq t_1 b_2 \\ & & \vdots \\ a_n t_n & \leq & a' \quad \quad s_n b_n \leq t_n b' \end{array}$$

for elements $a_i \in A$, $b_i \in B$, and $s_i, t_i \in S$.

An array such as that appearing in Proposition 1.2 is called a *scheme of length n from (a, b) to (a', b')* . It is useful to note that, in case $B = Sb \cup Sb'$, one may make the additional assumption that each b_i appearing in the right-hand sequence of inequalities is actually a member of $\{b, b'\}$.

2. FLATNESS PROPERTIES

As in the context of S -acts, there is a hierarchy of properties of S -posets that may be loosely termed *flatness properties*. These properties range, in decreasing strength, from *freeness*, already considered earlier, through various notions definable either in terms of tensor products or sometimes using so-called *interpolation properties*, to *torsion freeness*. The definitions generally bear strong resemblance to the corresponding unordered versions, and are often obtained by simply replacing $=$ by $\leq$ or $\geq$ in one or more places. But subtleties can arise along the way, often caused by the fact that, for S -posets, monomorphisms and embeddings are not the same. In this section, we will describe the most common properties that have been investigated to date. A few details are given below.

2.1. Projectivity. An S -poset P is called (*regular*) *projective* if, whenever A_S and B_S are S -posets and $\alpha : A \rightarrow B$ and $\beta : P \rightarrow B$ are S -poset maps with α a (regular) epimorphism, there exists an S -poset map $\gamma : P \rightarrow A$ such that $\alpha \circ \gamma = \beta$. Although it has been noted above that not all epimorphisms are regular, it has nonetheless been shown in [5] that projectivity and regular projectivity coincide. Furthermore, it was shown in [23] that, as is the case for S -acts, projective S -posets are exactly coproducts of families of S -posets each of the form eS where $e^2 = e \in S$. From this, it is clear that free S -posets (over sets) are projective, and in [5] it was proven that the free S -poset $P \times S$ over a poset P is projective if and only if P has the discrete order.

2.2. Strong flatness, and Conditions (E), (P) and (P_w). Among the first papers dealing with flatness properties of S -acts was [19], in which, by analogy with the Govorov-Lazard Theorem for R -modules, Stenström characterized the S -acts that are directed colimits of finitely-generated frees. An S -act A_S is called *pullback flat* (resp. *equalizer flat*) if the functor $A_S \otimes -$ (from the category $S\text{-Act}$ to the category Set) preserves pullbacks (resp. equalizers). Stenström also introduced two interpolation-type conditions, later called *condition (P)* and *condition (E)* by Peeter Normak. The situation here is that

$$\begin{aligned} \text{pullback flat} &\Rightarrow \text{condition (P)} \\ \text{equalizer flat} &\Rightarrow \text{condition (E)}, \end{aligned}$$

and, although neither of these implications is reversible, the conjunction of pullback flatness and equalizer flatness is the same as the conjunction of conditions (P) and (E). Acts satisfying all of these conditions are often nowadays called *strongly flat*. Interestingly, it was shown in [2] that, for S -acts, strong flatness and pullback flatness actually coincide (and so pullback flat implies equalizer flat, even though condition (P) and condition (E) are independent). In this paper, a single interpolation-type condition (PF) was introduced that characterizes strong flatness.

For S -posets conditions (P) and (E) are formulated as follows (see for example [4]). An S -poset A_S *satisfies property (P)* if, for all $a, a' \in A$ and $s, s' \in S$,

$$as \leq a's' \text{ implies } a = a''u, a' = a''u' \text{ for some } a'' \in A, u, u' \in S \\ \text{with } us \leq u's',$$

and it *satisfies property (E)* if, for all $a \in A$ and $s, s' \in S$,

$$as \leq as' \text{ implies } a = a''u \text{ for some } a'' \in A, u \in S \text{ with } us \leq us'.$$

In [4] it is shown that, as in the unordered situation, the S -posets A_S that satisfy conditions (P) and (E) are precisely those that are directed colimits of finitely generated free S -posets, and a connection with preservation properties of the functor $A_S \otimes -$ is given there as well. In [3] it has recently been shown that, in contrast to the situation for unordered acts, subpullback flatness does not imply subequalizer flatness. At the moment, it is unknown whether a single interpolation condition similar to condition (PF) exists that characterizes strongly flat S -posets.

A weaker form of condition (P), introduced in [21], that plays a role in our studies is the following: an S -poset A_S *satisfies condition (P_w)* if

$$as \leq a's' \text{ implies } a \leq a''u, a''u' \leq a' \text{ for some } a'' \in A, u, u' \in S \\ \text{with } us \leq u's'.$$

Examples given in [21] and [7] show that condition (P_w) does not imply condition (P). The observant reader will note that attempting by analogy to define a condition (E_w) does not lead to anything new.

An S -poset is called *strongly flat* if it satisfies both condition (P) and condition (E). In [4] other characterizations of strongly flat S -posets are given, in the spirit of [19], in terms of such concepts as pure epimorphisms, subpullbacks, and subequalizers. The interested reader is referred to the [4] for details. It is not difficult to show that projective S -acts are strongly flat, and that cyclic S -posets are strongly flat if and only if they satisfy condition (E).

Problem 1: Is there a single interpolation-type condition similar to condition (PF) for acts that characterizes strongly flat S -posets?

2.3. Flatness. An S -poset A_S is called *flat* if, for every S -poset ${}_S B$ and for all pairs $(a, b), (a', b')$ in $A \times B$, if $a \otimes b = a' \otimes b'$ in $A_S \otimes {}_S B$, then the same equality holds also in $A_S \otimes (Sb \cup Sb')$. (This means, equivalently, that the functor $A_S \otimes -$ takes embeddings in $S\text{-Pos}$ to monomorphisms in Pos .) A_S is called *weakly flat* (resp. *principally weakly flat*) if the functor $A_S \otimes -$ maps embeddings of (principal) left ideals $I \leq S$ to monomorphisms in Pos . (Recall that, for S -posets and for posets, the monomorphisms are the injective, monotonic morphisms, whereas the embeddings are the order-embeddings. Although the term *ideal* in the context of pomonoids often refers to a monoid ideal that is also an order ideal, unless otherwise specified we will not make that requirement in this article.) In terms of schemes, A_S is flat if and only if, for every $a, a' \in A$, every S -poset ${}_S B$, and every $b, b' \in B$, the existence of a scheme of the form

$$\begin{array}{rcccl} a & \leq & a_1 s_1 & & \\ a_1 t_1 & \leq & a_2 s_2 & & s_1 b \leq t_1 b_2 \\ & & \vdots & & \\ a_n t_n & \leq & a' & & s_n b_n \leq t_n b' \\ a' & \leq & a'_1 u_1 & & \\ a'_1 v_1 & \leq & a'_2 u_2 & & u_1 b' \leq v_1 b'_2 \\ & & \vdots & & \\ a'_m v_m & \leq & a & & u_m b'_m \leq v_m b \end{array}$$

implies that a replacement scheme (of possibly different length) can be found in which each b_i and each b'_i belongs to the set $\{b, b'\}$. For S -acts, it can be shown that condition (P) for an act A_S is equivalent to every tensor equality in every tensor product $A \otimes B$ being “witnessed” by a scheme of length 1.

Problem 2: Is the equivalence just mentioned also also true for S -posets?

Properties of *po-flatness* were introduced in [21] and correspond to whether the functor $A_S \otimes -$ preserves embeddings, rather than taking embeddings to monomorphisms. In other words, the problem becomes that of determining when $a \otimes b \leq a' \otimes b'$ in $A_S \otimes {}_S B$ implies that the same inequality holds also in $A_S \otimes (Sb \cup Sb')$. In [21] and [7] examples are given showing that each po-flatness property is strictly stronger than the corresponding flatness property. The following proposition in [21] gives a nice connection between principal weak po-flatness and weak po-flatness.

Proposition 2.1. *A right S -poset A_S is weakly po-flat if and only if it is principally weakly po-flat and satisfies the following Condition (W): for all $a, a' \in A$ and all $x, y \in S$*

$$\text{if } ax \leq a'y, \text{ then there exist } a'' \in A, p \in Sx, \text{ and } q \in Sy \text{ such that} \\ p \leq q, ax \leq a''p, \text{ and } a''q \leq a'y.$$

For completeness, we also note that Example 25 of [7] presents a flat S -poset A_S that is not mono-flat (that is, $A_S \otimes -$ does not preserve monomorphisms). Up to the present time, mono-flatness of S -posets has not been extensively studied.

2.4. Torsion freeness. An S -poset A_S is called *torsion free* if $ac = a'c'$ implies $a = a'$ whenever $a, a' \in A$ and c is a right cancellable element of S . An element c of a pomonoid S is called *right po-cancellable* if, for all $s, s' \in S$, $sc \leq s'c$ implies $s \leq s'$, and A_S is called *po-torsion free* if $ac \leq a'c$ implies $a \leq a'$ whenever $a, a' \in A$ and c is right po-cancellable. Right po-cancellable elements are right cancellable, but not conversely. Moreover, examples given in [7] show that the properties “torsion free” and “po-torsion free” are incomparable.

2.5. Summary diagram of flatness properties. The following relations exist among flatness properties of S -posets.

$$\begin{array}{ccccccc} \text{free} \Rightarrow \text{projective} \Rightarrow & (P) & & & & & \\ & \Downarrow & & & & & \\ & (P_w) & & & & & \\ & \Downarrow & & & & & \\ \text{po-flat} \Rightarrow \text{w. po-flat} \Rightarrow \text{p. w. po-flat} \Rightarrow & \text{po-t. free} & & & & & \\ & \Downarrow & & & & & \\ \text{flat} \Rightarrow \text{w. flat} \Rightarrow \text{p. w. flat} \Rightarrow & \text{t. free} & & & & & \end{array}$$

(incomparable)

All of the implications in the diagram above are strict, with the possible exception of $(P_w) \Rightarrow \text{po-flat}$, whose status is presently unknown.

3. SOME RESULTS

Having presented the main definitions pertinent to our discussion, we will now describe the general nature of the ongoing research in the area, giving as many specific results as we think interesting or appropriate.

3.1. Intrinsic descriptions of properties. In order to facilitate detailed study, we first try to give working “intrinsic” descriptions of the flatness properties we have introduced. This is relatively easy for uncomplicated types of S -posets, but is probably not possible in general.

3.1.1. One-element S -posets. The theorem below [7] gives a complete description of when a one-element S -poset Θ_S has each property.

Theorem 3.1. *Let S be any pomonoid. Then:*

- (1) Θ_S is free if, and only if, $|S| = 1$.
- (2) Θ_S is projective if, and only if, S has a left zero element.
- (3) Θ_S satisfies condition (E) if, and only if, S is left collapsible.
- (4) The following assertions are equivalent:
 - (a) Θ_S satisfies condition (P);

- (b) Θ_S satisfies condition (P_w) ;
 - (c) Θ_S is po-flat;
 - (d) Θ_S is flat;
 - (e) Θ_S is weakly po-flat;
 - (f) Θ_S is weakly flat.
 - (g) S is weakly right reversible.
- (5) Θ_S is (always) principally weakly (po-) flat and (po-)torsion free.

(We recall that a monoid is called right reversible if $Ss \cap St \neq \emptyset$ for all s and t , left collapsible if for every s and t there exists u such that $us = ut$, and weakly right reversible if $Ss \cap (St] \neq \emptyset$ for all s and t . Here, $(St]$ denotes the set of all elements p of S such that $p \leq vt$ for some $v \in S$. An example of a weakly right reversible pomonoid that is not right reversible is given in [7].)

3.1.2. Rees factor S -posets. Suppose K_S is a proper right ideal of a pomonoid S . One can show that, in order for the factor right S -poset $S/\theta(K \times K)$ (defined earlier) to coincide (as an S -act) with the usual Rees factor act (in which K is the only non-trivial congruence class), it is necessary and sufficient that K be a convex subset of S as well as a right ideal. If K_S is a convex, proper right ideal of the pomonoid S , S/K_S will always stand for $S/\theta(K \times K)$, with the order introduced earlier (and described explicitly below). For $s \in S$, the congruence class of s in S/K_S will be denoted $[s]$. So far, in [7], we have been able to give intrinsic descriptions of most of the flatness properties for S/K_S , but some exceptions will be mentioned as a Problem at the end of this section.

Lemma 3.2. *Let K_S be a convex, proper right ideal of the pomonoid S . Then for $x, y \in S$,*

$$[x] \leq [y] \text{ in } S/K_S \Leftrightarrow (x \leq y) \text{ or } (x \leq k \text{ and } k' \leq y \text{ for some } k, k' \in K).$$

Moreover, $[x] = [y]$ in S/K_S if, and only if, either $x = y$ or else $x, y \in K$.

Lemma 3.3. *Let K_S be a convex, proper right ideal of the pomonoid S , let ${}_S M$ be any left S -poset, and let $m, m' \in M$. Then $[1] \otimes m \leq [1] \otimes m'$ in $S/K_S \otimes {}_S M$ if, and only if, either $m \leq m'$, or*

$$\begin{array}{rcl} m & \leq & k_1 m_1 \\ k'_1 m_1 & \leq & k_2 m_2 \\ & \vdots & \\ k'_n m_n & \leq & m' \end{array}$$

for elements k_i, k'_i of K and m_i of M .

Using the foregoing results one can prove the following theorem [7]:

Theorem 3.4. *Let K_S be a convex, proper right ideal of the pomonoid S . Then:*

(1) *S/K_S is torsion free if, and only if, for every $s \in S$ and every right cancellable $c \in S$,*

$$sc \in K \text{ implies } s \in K.$$

(2) *S/K_S is po-torsion free if, and only if, whenever c is a right po-cancellable element of S such that $sc \leq k$ and $l \leq tc$ for elements $s, t \in S$ and $k, l \in K$, there exist $k', l' \in K$ such that $s \leq k'$ and $l' \leq t$.*

(3) *S/K_S is principally weakly flat if, and only if, for every $k \in K$ there exist $k', k'' \in K$ such that $k'k \leq k \leq k''k$.*

(4) S/K_S is principally weakly po-flat if, and only if, for every $k \in K$ and $s \in S$,

$$\begin{aligned} k \leq s &\Rightarrow (\exists k' \in K)(k's \leq s) \text{ and} \\ s \leq k &\Rightarrow (\exists k' \in K)(s \leq k's). \end{aligned}$$

(5) S/K_S is weakly flat if, and only if (i) S/K_S is principally weakly flat, and, (ii) S is weakly right reversible.

(6) S/K_S is weakly po-flat if, and only if, (i) S/K_S is principally weakly po-flat, and (ii) S is weakly right reversible.

(7) S/K_S has property (P) if, and only if, $|K| = 1$.

The following question remains open.

Problem 3: What conditions on a pomonoid S and a convex, proper right ideal K_S guarantee that S/K_S is flat, po-flat, or has property (P_w)?

3.1.3. Cyclic S -posets. Some of the results of the preceding sections can be extended to general cyclic S -posets S/ρ where ρ is a right congruence on S (that is, an S -poset congruence on S_S). For (unordered) S -acts, characterizations of flatness of S/ρ in terms of left congruences on S were developed by V. Fleischer [11]. A main result of that work was that, if the functor $A_S \otimes -$ preserves all embeddings of left S -acts into cyclic acts, then A_S is flat. The same approach has been initiated in [22] for S -posets, and although it has not yet been completely successful, some useful partial results have been obtained that we now briefly discuss. In the following, if λ and ρ are left and right S -poset congruences, respectively, on a pomonoid S , then relations $\hat{\lambda}$ and $\hat{\rho}$ are defined by

$$\begin{aligned} s\hat{\lambda}t &\Leftrightarrow s\lambda \leq t\lambda \text{ in } S/\lambda, \text{ and} \\ s\hat{\rho}t &\Leftrightarrow s\rho \leq t\rho \text{ in } S/\rho. \end{aligned}$$

Let $\hat{\lambda} \sqcup \hat{\rho}$ denote the transitive closure of $\hat{\lambda} \cup \hat{\rho}$.

Lemma 3.5. *Let ρ and λ denote right and left congruences, respectively, on a pomonoid S . Then $s\rho \otimes t\lambda \leq s'\rho \otimes t'\lambda$ in $S/\rho \otimes S/\lambda$ for $s, s', t, t' \in S$ if and only if $(st, s't') \in \hat{\lambda} \sqcup \hat{\rho}$.*

Lemma 3.6. *Let ρ denote a right congruence on the pomonoid S and let s belong to S . Then $u\rho \otimes s \leq v\rho \otimes s$ in $S/\rho \otimes S_S$ for $u, v \in S$ if and only if $(u, v) \in \hat{\rho} \sqcup \overrightarrow{\ker \rho_s}$.*

Using the lemmas above, the following proposition is proved in [22].

Proposition 3.7. *Let ρ be a right congruence on the pomonoid S . Then S/ρ is*

(1) *principally weakly po-flat if and only if, for $u, v, s \in S$,*

$$(us)\rho \leq (vs)\rho \text{ implies } (u, v) \in \hat{\rho} \sqcup \overrightarrow{\ker \rho_s};$$

(2) *principally weakly flat if and only if, for $u, v, s \in S$,*

$$(us)\rho = (vs)\rho \text{ implies } (u, v) \in \hat{\rho} \sqcup \overrightarrow{\ker \rho_s}; \text{ and}$$

(3) *weakly po-flat if and only if, for $s, t \in S$, $s\rho \leq t\rho$ implies that there exist $u, v \in S$ such that $us \leq vt$, $(u, 1) \in \hat{\rho} \sqcup \overrightarrow{\ker \rho_s}$, and $(1, v) \in \hat{\rho} \sqcup \overrightarrow{\ker \rho_t}$.*

The following problem is posed in [22]:

Problem 4: What conditions on a pomonoid S and a right congruence ρ on S guarantee that S/ρ is flat, po-flat, weakly flat, has property (P_w) , or is po-torsion free?

3.2. Homological classification. Historically, the study of flatness properties (whether it be of modules over a ring or acts over a monoid) has been concerned with two types of questions, namely, when all acts (or modules) have a given flatness property, and when two generally-distinct flatness properties actually coincide. As an example of the first type, all modules over a ring R are free if and only if R is a division ring, and all acts over a monoid S satisfy condition (P) if and only if S is a group. To illustrate the second, an important result due to Bass states that all flat left modules over a ring R are projective if and only if the ring is perfect (that is, it has the minimum condition on principal right ideals), whereas for acts, all flat right S -acts are projective if and only if $|S| = 1$. Below, we give an idea of what is known about the corresponding questions for S -posets.

3.2.1. When all S -posets have a particular property. In [13] already, it was shown that pogroups are absolutely (po-)flat (meaning that, if S is a partially ordered group, then all left and right S -posets are (po-)flat). This result was re-proved in slightly more elegant fashion in [23], and the sharper result that every S -poset over a pogroup S satisfies condition (P_w) was proved in [21]. In contrast to the situation for S -acts (mentioned above), it was noted in [23] that there exists *no* pomonoid S over which all A_S satisfy condition (P). To see this, simply let S act trivially on a two-element chain $x < y$. Then $xs \leq ys$ but there exist no z, u, v such that $zu = x$, $zv = y$, and $us = vs$. The rather obvious conjecture that S must be a pogroup in order for all S -posets to satisfy condition (P_w) is defeated by the following proposition in [7]:

Proposition 3.8. *Let $S = G^0$ be an ordered group with zero, in which 0 is either the largest or the smallest element of S . Then every S -poset satisfies condition (P_w) (and so, in particular, S is absolutely po-flat).*

For S -acts it is well known that all A_S are principally weakly flat if and only if S is a regular monoid. For a pomonoid S , if S is regular it is easy to show that every S -poset is principally weakly flat. That the converse is also true is proved in [18], in which the authors show that, if I is a proper right ideal of a pomonoid S and $A(I)$ is principally weakly flat, then for every $i \in I$ there exists $j \in I$ such that $i = ji$. From this the regularity of S follows easily.

In [12] it was shown that inverse monoids are absolutely flat. The same result was proved, independently, in [6], where a characterization of the left absolutely flat completely (0-)simple semigroups was given. The following S -poset results from [7] contrast sharply with the S -act results just mentioned.

Proposition 3.9. (1) *Let $S = (\mathcal{M}(G; I, \Lambda; P))^1$ be a completely simple semigroup (represented as a Rees matrix semigroup) with adjoined 1 such that the group G is periodic. Then a pomonoid $(S, \leq)$ is right absolutely flat if and only if $|\Lambda| = 1$; that is, S is a left group with adjoined 1.*

(2) *If G has an element of infinite order and I is a left zero semigroup containing at least two distinct elements, then the pomonoid $S = (I \times G)^1$, with the discrete order, is not right absolutely flat.*

(3) Let S be a pomonoid in which 1 is the largest element, and suppose there exist $\beta < \alpha < 1$ in S such that $\alpha \notin S\beta$. Then $A/(\alpha S]$ is not flat.

It follows from (3) above that a semilattice with 1, considered as a pomonoid with its natural order, is absolutely flat if and only if it has at most two elements. In fact, among idempotent monoids having 1 as largest element, it can be shown that only the 1- or 2-element semilattice is right absolutely flat. Other absolute flatness results for pomonoids are presented in [7], but many open questions remain in this area.

3.2.2. *When generally-distinct properties coincide.* In [22] the following result appears.

Proposition 3.10. *The following conditions on a pomonoid S are equivalent:*

- (1) *all projective right S -posets are free;*
- (2) *all cyclic projective right S -posets are free;*
- (3) *all cyclic strongly flat right S -posets are free;*
- (4) *for every $e^2 = e \in S$, $e\mathcal{D}1$ (where $\mathcal{D}$ denotes Green's relation).*

Otherwise, to the author's knowledge, no other work has to date appeared in this area.

REFERENCES

- [1] J. Adamek, H. Herrlich, G. Strecker, *Abstract and Concrete Categories*, available on-line at <http://katmat.math.uni-bremen.de/acc/acc.pdf>.
- [2] S. Bulman-Fleming, *Pullback flat acts are strongly flat*, Canad. Math. Bull. **34** (1991), 456–461.
- [3] S. Bulman-Fleming, *Subpullback flat S -posets need not be subequalizer flat*, Semigroup Forum (online publication 2008).
- [4] S. Bulman-Fleming, V. Laan, *Lazard's theorem for S -posets*, Math. Nachr. **278** (2005), 1–13.
- [5] S. Bulman-Fleming, M. Mahmoudi, *The category of S -posets*, Semigroup Forum **71** (2005), 443–461.
- [6] S. Bulman-Fleming, K. McDowell, *Absolutely flat semigroups*, Pac. J. Math. **107** (1983), 319–333.
- [7] S. Bulman-Fleming, D. Gutermuth, A. Gilmour, M. Kilp, *Flatness properties of S -posets*, Comm. Alg. **34** (2006), 1291–1317.
- [8] T. Blyth, M.F. Janowitz, *Residuation Theory*, Pergamon, Oxford, 1972.
- [9] S.L. Bloom, *Varieties of ordered algebras*, J. Comput. System Sci. **13** (1976), 200–212.
- [10] G. Czédli, A. Lenkehegyi, *On classes of ordered algebras and quasiorder distributivity*, Acta. Sci. Math. **46** (1983), 41–54.
- [11] V. Fleischer, *Flat relative to diagram acts* (Russian), Summaries of the conference Theoretical and Applied Problems of Mathematics, 17–19, Tartu, 1980.
- [12] V. Fleischer, *Completely flat monoids* (Russian), Tartu ÜL Toimetised **610** (1982), 38–52. AMS Translations, Series 2 **142** (1989), 19–32.
- [13] S.M. Fakhruddin, *Absolute flatness and amalgams in pomonoids*, Semigroup Forum **33** (1986), 15–22.
- [14] S.M. Fakhruddin, *On the category of S -posets*, Acta Sci. Math. (Szeged) **52** (1988), 85–92.
- [15] J.M. Howie, *Fundamentals of Semigroup Theory*, Clarendon, Oxford, 1995.
- [16] M. Kilp, U. Knauer, A. Mikhalev, *Monoids, Acts and Categories*, Walter de Gruyter, Berlin, 2000.
- [17] N. Kehayopulu, M. Tsingelis, *On subdirectly irreducible ordered semigroups*, Semigroup Forum **50** (1995), 161–177.
- [18] H. Qiao, F. Li, *When all S -posets are principally weakly flat*, Semigroup Forum **75** (2007), 536–542.
- [19] B. Stenström, *Flatness and localization over monoids*, Math. Nachr. **48** (1971), 315–333.

- [20] X. Shi, W. Hu, *Indecomposable, projective unitary S -posets*, Southeast Asian Bull. Math. **30** (2006), 943–949.
- [21] X. Shi, *Strongly flat and po-flat S -posets*, Comm. Alg. **33** (2005), 4515–4531.
- [22] X. Shi, *On flatness properties of cyclic S -posets*, Semigroup Forum (online publication 2007).
- [23] X. Shi, Z. Liu, F. Wang, S. Bulman-Fleming, *Indecomposable, projective and flat S -posets*, Comm. Alg. **33** (2005), 235–251.
- [24] X.Y. Xie, X.P. Shi, *Order-congruences on S -posets*, Comm. Korean Math. Soc. **20** (2005), 1–14.

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