

MORE ON UNRETRACTIVE GRAPHS

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ABSTRACT. One of the established ways of doing research in algebraic graph theory involves groups. In particular, automorphisms of graphs have been studied intensively. One of the related problems is to characterize the graphs whose endomorphism monoid is their automorphism group. In a forthcoming paper that problem was solved but the focus of discussion has been on lexicographic products of graphs. In this paper an improved characterization for the general case is provided.

1. INTRODUCTION

Studying automorphism groups of graphs dates back at least to Pólya [25], Frucht, [5], and Zykov, [29]. The latter authors studied automorphism groups of special cases of the lexicographic product namely $\overline{K_n}[H]$ and $K_2[H]$, respectively. A characterization of those graphs G, H for which $\text{Aut } G[H] = \text{Aut } G \wr \text{Aut } H$ was given by Sabidussi in [26] and then generalized to a larger class of pairs of graphs by Hemminger in [13], [14]. Hahn extended Sabidussi's results to hypergraphs [7]. In group theory since long, see e.g. [28, p. III], homomorphisms are used for achieving an overview over the variety of existing groups. That idea is applied in this paper too. For many graphs G there is an abundance of endomorphisms. The rational way of using endomorphisms for getting an overview over the subgraphs of a graph thus is using the kernels of graph endomorphisms.

The study of unretractive graphs belongs to the wider area of studying graph retractions that was pioneered by Hell [9, 10]. Hell's motivation was to understand extension of graph morphism from subgraphs to supergraphs. Unretractive graphs, using a different terminology, were studied by Hell, as well as by Hell and Nešetřil already in [10] and [11], respectively. Knauer has researched unretractive graphs in [20, 21, 22], i.e. aimed at characterizing those graphs G for which $\text{End } G = \text{Aut } G$ holds. Related work was also published in [17] and a number of related comments is available in [8] as well as in [6]. In [12] it is shown that graphs with independence number two are unretractive iff their complement is bi-critical. In [4] it is shown that the size of a largest clique in an unretractive graph is below half the sum of the number of vertices of that graph and the number of its central vertices.

A number of results concerning unretractive graphs was announced in [16]. These are proven in [18]. In the latter paper unretractivity of graphs has been characterized in terms of congruences. That line of research is continued here and an improved characterization is provided.

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2. PRELIMINARIES

The image of a mapping $f : S \rightarrow T$ is denoted by $im(f)$. The cardinality of a set S and the identity mapping on it are denoted by $|S|$ and 1_S , respectively. It is well-known that for each element s of a finite semigroup S there exists a positive integer n such that s^n is idempotent, i.e., $s^n s^n = s^n$.

In this paper a **graph** G is a pair (VG, EG) of a finite vertex set VG and an edge set EG , whose elements are two element subsets of VG . Rather than $\{g, g'\} \in EG$ we are going to write $gg' \in EG$. If G is a graph with $|VG| = n$ then it is denoted by K_n, D_n, P_n, C_n , and W_n if it is **complete**, **discrete**¹, a **path**, a **cycle**, or a **wheel** of order n , respectively. A graph S is called **partgraph** of a graph G if $VS \subseteq VG$ and $ES \subseteq EG$. For a partgraph S of a graph G the graph $\langle S \rangle_G = (VS, E)$ with $E = \{\{s, s'\} \subseteq VS \mid ss' \in EG\}$ is called the subgraph **induced** by S . Obviously, $\langle (VS, \emptyset) \rangle_G = \langle S \rangle_G$, for each partgraph S of G . One usually only considers subgraphs $\langle VS \rangle_G$ induced by subsets VS of the vertex set VG of a graph G . In algebraic papers often the sole considered substructure type is that of induced subgraph.

Let G, H be graphs. A **graph morphism** $r : G \rightarrow H$ is an edge preserving vertex mapping. The mapping it induces on the edges is denoted by r_E . The set of graph morphisms from G into H is denoted by $Hom(G, H)$. The set $Hom(G, G)$ is denoted by $End\ G$ and each of its elements is called an **endomorphism**. We are going to denote 1_{VG} by 1_G . A graph H is called **retract** of a graph G if there are morphisms $\iota : H \rightarrow G$ and $\pi : G \rightarrow H$ such that $\pi \circ \iota = 1_H$. In that case π and ι are called **retraction** and **co-retraction**, respectively. The graph G is also called **co-retract** of H and H is called **retract** of G . A morphism that is retraction and co-retraction is called **isomorphism**.

The **chromatic number** $\chi(G)$ of a graph G can be defined as $\min \{n \mid Hom(G, K_n) \neq \emptyset\}$. One of the equivalent definitions of bipartite graph is that its chromatic number is at most two. Corollary 8 of [19] states that $\chi(G[H]) \geq \chi(G) + 2\chi(H) - 2$, for graphs G, H with $EG \neq \emptyset$. For an odd cycle $\mathcal{O}$ and a bipartite graph $\mathcal{B}$ with $EB \neq \emptyset$, that corollary together with $\chi(\mathcal{O}) = 3$ and $\chi(\mathcal{B}) = 2$ entails that $\chi(\mathcal{O}[H]) > 2\chi(H)$ and $\chi(\mathcal{B}[H]) = 2\chi(H)$. Obviously, $\chi(H) \geq \chi(G)$ if there exists a morphism $\phi : G \rightarrow H$. The graph G is **color critical** if $\chi(H) < \chi(G)$ for each induced proper subgraph H of G .

An isomorphism $\phi \in End\ G$ is called **automorphism** and the automorphism group of G is denoted by $Aut\ G$. It is well-known and easy to show that the automorphisms of a graph are exactly the bijective endomorphisms of that graph. A graph G is called **unretractive** if $End\ G = Aut\ G$. In [8, 12], and [17, Theorem II.1.5] it has been shown that any two unretractive retracts of a given graph are isomorphic to each other. Each instance of that isomorphism class of a graph G is denoted by C_G and called the **core** of G . An unretractive graph, following [12], is usually called a core graph or simply a core. That term, however, is also used in different meanings, for example in [23, 24] and in [2]. Graphs can be considered as those irreflexive and symmetric structures over a signature with one relation symbol only that, moreover, has the arity 2. The concept of core thus canonically extends to structures. An example was given in [1, Theorem 6] of an infinite, directed graph G without proper retracts and $End\ G \neq Aut\ G$. That paper also

¹Recall, that discrete means edge-less.

shows (Theorem 9) that an infinite directed graph may have a proper retract and a retract that has no proper retracts. That paper claims, without proof, that these (and other) results also apply to undirected, infinite graphs.

The **identity relation** on a set S will be denoted by Δ_S (or even Δ). The **lexicographic product** of graphs G and H is denoted by $G[H]$. The generalized lexicographic product of G with a family of graphs $\{H_i\}_{1 \leq i \leq |VG|}$ is denoted by $G[H_1, \dots, H_{|VG|}]$ or $G[H_g \mid g \in VG]$. It in general is called join and was called lexicographic sum in [15].

An equivalence relation on the vertex set VG of a graph G is called **congruence** of G if each of its classes is an independent set. The **factor graph** F of G factored after a congruence c has the vertex set $VF = VG/c$ and the edge set $EF = \{\{f, f'\} \subseteq VF \mid f \neq f' \exists g \in f, g' \in f' \text{ with } gg' \in EG\}$. That factor graph will often be denoted by G/c . It is well-known that the **canonical mapping** $\kappa_c : G \rightarrow F$, $g \mapsto [g]_c$ is a graph morphism. The **kernel** $\ker(r)$ of a graph morphism $r : G \rightarrow H$ is the congruence $\{(g, g') \in VG \times VG \mid r(g) = r(g')\}$. A congruence c of G is called **internal** if there exist an induced subgraph G' of G and an isomorphism $\iota : G/c \rightarrow G'$. Otherwise c is called external. Using the terminology of Hahn and Tardif (see [8, Definition 2.2]) internal congruences are exactly the kernels of faithful endomorphisms. A congruence of G is called **retractance** (in [18] the term co-Rees congruence was used instead) if it is the kernel of an idempotent endomorphism of G . Each retraction of a graph obviously right-factors an idempotent endomorphism of that graph with the left factor being injective. So retractances are exactly the kernels of retractions. It is easy to see that a congruence c of G is a retractance iff for all $X \in VG/c$ there exists $g_X \in X$ such that $g_X g_Y \in EG$ if there exist $\xi \in X, \eta \in Y$ such that $\xi \eta \in EG$. Obviously, $\text{im}(r) \cap X$ is a singleton set for $r^2 = r \in \text{End } G$ and $X \in VG/\ker(r)$. Clearly $\{g_X\} = X \cap \text{im}(r)$, for all $X \in VG/\ker(r)$. Let c, c' be congruences of G then c is called **refinement** of c' if each c -class is contained in a c' -class.

3. UNRETRACTIVE GRAPHS

Definition 3.1. A surjective morphism $\phi \in \text{Hom}(G, H)$ is called **glue** if it has exactly one non-singleton-set kernel class S_ϕ , that actually has cardinality 2, and ϕ_E is surjective. A glue ϕ is called **simple fold** if the vertices in S_ϕ have a common neighbor. Below S_ϕ is occasionally called **semi hinge**.

The term simple fold was borrowed from [8]. In that paper a glue is called an elementary morphism.

Example 3.2. The example provides an internal congruence that has an external refinement. Let G be the six-cycle with $VG = \{a, b, c, d, e, f\}$ and $EG = \{ab, bc, de, ef, fa\}$. Let furthermore H be the graph with $VH = \{a, b, \{c, e\}, d, f\}$ and $EH = \{ab, b\{c, e\}, \{c, e\}d, \{c, e\}f, fa\}$. Then

$$\phi : VG \rightarrow VH, \quad g \mapsto \begin{cases} x, & \text{if } x \notin \{c, e\} \\ \{c, e\}, & \text{else} \end{cases}$$

is a simple fold with $S_\phi = \{c, e\}$. The relation $\ker(\phi)$ is an external congruence since G does not contain an induced four-cycle. It refines, however, the internal

congruence $\ker(\psi)$ with ψ being defined as

$$\psi : VG \rightarrow VI, \quad g \mapsto \begin{cases} g, & \text{if } g \notin \{b, f\} \cup S_\phi \\ b, & \text{if } g \in \{b, f\} \\ e, & \text{otherwise.} \end{cases}$$

Note that $\ker(\psi)$ is a retractance with the classes $\{a\}$, $\{b, f\}$, $\{c, e\}$, and $\{d\}$. As a system of distinguished representatives one needs to choose $\{a, f, e, d\}$.

Example 3.3. Obviously, for unretractive graphs any internal congruence is a retractance. It is easy to show that an internal congruence c of a graph G is a retractance iff there is one copy of G/c in G that intersects each c -class in exactly one vertex. This example shows that that condition in general is not true and that thus an internal congruence not necessarily is a retractance. Let $G = (\{a, b, c, d, e, f, g\}, \{af, ag, ge, fe, fd, cd, ba\})$. Consider the congruence $c = \{\{a\}, \{b, c\}, \{d, e\}, \{g, f\}\}$ on G . The factor graph G/c is isomorphic to C_4 . That graph is an induced subgraph of G and so c is an internal congruence. It, however, is not a retractance. Assume otherwise. Then a $\phi^2 = \phi \in \text{End } G$ exists such that $c = \ker(\phi)$. Since a 's c -class is a singleton set we have $\phi(a) = a$. Now $(b, g), (b, f) \notin \ker(\phi)$ and so $\phi(g) \neq b \neq \phi(f)$. Therefore $\phi(\{g, f\}) \subseteq \{g, f\}$. Assume now $\phi(f) = g$. Then $\phi(g) = g$, because of $(g, f) \in \ker(\phi)$. This implies $\phi(b) \neq g$, because $(g, b) \notin \ker(\phi)$. Consequently $\phi(b) = f$. But then ϕ is not idempotent because of $\phi(f) = g \neq f$. Therefore $\phi(f) = f$ and $\phi(g) = f$. Then $\phi(b) \neq f$, because of $(b, f) \notin \ker(\phi)$. Thus $\phi(b) = g$ and again ϕ is not idempotent, as $\phi(g) = f \neq g$.

We recall and paraphrase the proposition [8, Proposition 2.19] of Hahn and Tardif. A less compact proof of it is in [6, p. 114, 115]. Recall that the Proposition 3.4 does not extend to disconnected graphs. Recall too, that for any morphism $\phi : G \rightarrow H$ there is a sequence $\phi_1, \phi_2, \dots, \phi_n$ of glues such that $\tilde{\phi} = \phi_n \phi_{n-1} \dots \phi_1$, if one defines $\tilde{\phi} : G \rightarrow (\text{im}(\phi), \phi_E(EG))$, $g \mapsto \phi(g)$. Exactly the surjective morphisms that induce a surjective edge mapping are sequences of glues.

Proposition 3.4. *For a connected graph G any non-injective retraction $\pi \in \text{Hom}(G, H)$ is a sequence of simple folds, i.e., for any such retraction an integer n and a sequence $\Phi = \phi_n, \phi_{n-1}, \dots, \phi_1$ of simple folds exists such that $\pi = \phi_n \phi_{n-1} \dots \phi_1$.*

Proposition 3.5. *A connected graph G is unretractive iff the following assertions hold:*

- (i) *No simple fold $\phi \in \text{Hom}(G, H)$ has an internal congruence.*
- (ii) *Refinements of internal congruences of G are internal.*

Proof. Necessity. Let G be unretractive. Assume that a simple fold $\phi \in \text{Hom}(G, H)$ exists with internal $c = \ker(\phi)$. Then there is an induced subgraph G' of G and an isomorphism $\iota : G/c \rightarrow G'$. Since G/c is isomorphic to H we can assume that $\iota\phi \in \text{End } G \setminus \{1_G\}$. Thus an integer n exists such that $(\iota\phi)^n(\iota\phi)^n = (\iota\phi)^n \neq 1_G$ and $\text{End } G \neq \text{Aut } G$ and G is retractive. This implies (i). Assume now that an internal congruence c of G exists that has an external refinement c' . Then $c' \neq \Delta_G$. Now c is a refinement of the kernel $\ker(\phi)$ of an idempotent endomorphism $\phi^2 = \phi \in \text{End } G$. Since G is unretractive $\phi = 1_G$ and so $\ker(\phi) = \Delta_G$ which is impossible because c' refines c and itself is different from Δ_G .

Sufficiency. Let the assertions (i) and (ii) be true. Assume that G is retractive, then $\text{End } G \setminus \text{Aut } G \neq \emptyset$ and an idempotent $\phi^2 = \phi \in \text{End } G \setminus \{1_G\}$ exists. Now $\phi' : VG \rightarrow \text{im}(\phi)$, $g \mapsto \phi(g)$ is a retraction. Because of Proposition 3.4 there is a sequence $\Phi = \phi_n, \phi_{n-1}, \dots, \phi_1$ of simple folds such that $\phi' = \phi_n \phi_{n-1} \dots \phi_1$. Because of (i) $\ker(\phi_1)$ is external and because of $\phi \in \text{End } G$ the congruence $\ker(\phi) = \ker(\phi')$ is internal. Now $\ker(\phi_1)$ is an external refinement of $\ker(\phi)$ and this contradicts (ii). Thus G is retractive. $\square$

Remark 3.6. Neither of the assertions of Proposition 3.5 implies the other.

Proof. Simple folds of the C_6 do not have an internal congruence. The example above shows that C_6 has internal congruences that are refined by external congruences. So (i) does not imply (ii). Consider the path P_3 . It obviously has exactly two congruences both of which are internal. Therefore (ii) applies to P_3 . However, (i) does not, as each morphism having the non-trivial congruence as kernel is a simple fold and has an internal congruence. So (ii) does not imply (i). $\square$

We add a characterization of unretractive graphs involving glues rather than simple folds.

Proposition 3.7. *A graph G is unretractive iff the following assertions hold:*

- (i) *No glue $\phi \in \text{Hom}(G, H)$ has an internal congruence.*
- (ii) *Refinements of internal congruences of G are internal.*

Proof. It suffices to show the condition's sufficiency. Assume that G is retractive. Then it has a non injective endomorphism $\phi \in \text{End } G$. There exists an integer n such that $\phi^n \phi^n = \phi^n$ and the target restricted morphism $\tilde{\phi}^n : G \rightarrow \langle \text{im}(\phi^n) \rangle_G$ is a sequence of glues. Therefore the kernel $\ker(\phi^n)$ of ϕ^n is refined by the kernel c of a glue. That c , because of (ii) and contrary to (i), is an internal congruence. $\square$

The example of P_3 shows that absence of non-simple-fold glues with internal kernel does not imply the absence of simple folds with internal kernel. The following definition of dunking essentially is a paraphrase from von Braunschweig [27]. The concept of dunking is a generalization of the well-known concept of interval, see for example [15]. In [3] it has been referred to as externally related vertex set. If gg' is an edge then occasionally a dunking (g, g') is called a corner. Keep in mind that the relation $\{(g, g') \in VG \times VG \mid (g, g') \text{ is a dunking}\}$ in general is not symmetric.

Definition 3.8. A pair (g, g') of vertices of graph G is called **dunking** if $g \neq g'$ and each neighbor of g is a neighbor of g' . G is called **dunking-free** if it has no dunkings.

Proposition 3.9. *For any graph G the following assertions are equivalent:*

- (i) *The graph G is dunking-free.*
- (ii) *No simple fold of G has an internal congruence.*

Proof. It is sufficient to show that (i) implies (ii). Let G be dunking-free and $\phi \in \text{Hom}(G, H)$ be a simple fold with semi-hinge $S_\phi = \{a, b\}$. Assume that $c = \ker(\phi)$ is internal. Then there exists an induced subgraph H' of G and an isomorphism $\iota : H \rightarrow H'$. Define the mapping

$$\sigma : VG \rightarrow VG, g \mapsto \iota\phi(g).$$

Then $\sigma \in \text{End } G$ and thus an integer n exists such that $\sigma^n \sigma^n = \sigma^n$. Now $\ker(\sigma) = \ker(\phi) = c$. For $g, g' \in VG$ the equation $\sigma^2(g) = \sigma^2(g')$ entails $\phi\iota\phi(g) = \phi\iota\phi(g')$. This implies $\iota\phi(g) = \iota\phi(g')$ and thus $\ker(\sigma^2) = \ker(\sigma)$ and consequently $\ker(\sigma^n) = \ker(\sigma)$. Thus $|\text{im}(\sigma^n)| = |VG| - 1$. Without loss of generality we can assume that $a \notin VH' = \text{im}(\sigma) = \text{im}(\sigma^n)$. Now σ^n is idempotent and $b \in \text{im}(\sigma^n)$. Thus $\sigma^n(b) = b$. Since $(a, b) \in c = \ker(\sigma^n)$ we have $\sigma^n(a) = \sigma^n(b)$ and (a, b) is a dunking. $\square$

Together with the observation that $\text{Hom}(C, C') = \emptyset$, for any two different components of an unretractive disconnected graph, the Propositions 3.9 and 3.5 imply:

Proposition 3.10. *A graph G is unretractive iff the following assertions hold:*

- (i) *it is dunking-free;*
- (ii) *any refinement of any of its internal congruences is internal; and*
- (iii) *there are no morphisms of any of its components into another one.*

Example 3.11. Retractances are internal congruences. They have a special internal structure. It would thus be interesting to know whether or not in the preceding proposition (ii) can be replaced by (ii)', i.e., that any refinement of any internal congruence is a retractance. Certainly (ii)' is necessary for unretractive graphs. The resulting condition (i), (ii)', and (iii) is, however, not sufficient for unretractivity. Consider again the C_6 . Recall that it has an internal congruence that is refined by an external congruence. So assertion (ii) does not apply to C_6 . However, assertion (ii)' does. Showing this is not difficult. It only requires having an overview over the congruences of C_6 . Such an overview can be obtained by listing the congruences of C_6 according to the pattern of cardinality of their classes. There are seven such patterns, i.e. 3/3, 3/1/2, 3/1/1/1, 2/2/2, 2/2/1/1, 2/1/1/1/1, and 1/1/1/1/1/1. For each of these patterns the essentially different congruences easily can be enumerated if one observes the symmetry of C_6 . It turns out that all the internal congruences are kernels of idempotent endomorphisms of C_6 .

Corollary 3.12. (i) (Knauer, [22]) *The graph $K_n[H_1, \dots, H_n]$ is unretractive iff $H_1, \dots, H_n$ are unretractive.*²

(ii) (Knauer, [20]) *The graph $C_{2n+1}[H]$ is unretractive iff H is unretractive.*

Proof. The trivial proof for (i) is omitted. The proof for (ii) follows from the observation that any internal congruence c of $C_{2n+1}[H]$ can have no class Γ that intersects $g[H]$ and $g'[H]$ for $g, g' \in VC_{2n+1}$ with $g \neq g'$ because otherwise each induced subgraph K of $C_{2n+1}[H]$ that is isomorphic to $C_{2n+1}[H]/_c$ would contain a cycle of length shorter than $2n+1$ of vertices each of which have a degree of at least $2|VH|$ in $C_{2n+1}[H]$. Clearly no such cycle can exist in $C_{2n+1}[H]$. That observation implies that $C_{2n+1}[H]$ is dunking free and that each of its internal congruences is Δ if H is unretractive. $\square$

Proposition 3.13. *Let G, H be graphs and $\phi : G \rightarrow H$ a glue with semi-hinge $S_\phi = \{a, b\}$. Then $\chi(G) < \chi(H)$ iff $\gamma(a) \neq \gamma(b)$ holds for all $\gamma \in \text{Hom}(G, K_{\chi(G)})$.*³

Proof. Obviously, H is isomorphic to $G/\ker(\phi)$, compare, e.g. [8, Proposition 2.12]. Therefore we can assume $H = G/\ker(\phi)$ and that ϕ is the canonical morphism with

²For $n = 2$ this is shown in [4].

³I thank the reviewer for pointing out that this proposition applies to glues rather than only to simple folds.

respect to $\ker(\phi)$. Denote $\chi(G)$ by c . Let first be $c < \chi(H)$. Assume a morphism $\gamma : G \rightarrow K_c$ exists with $\gamma(a) = \gamma(b)$. Define

$$\gamma' : VH \rightarrow VK_c, \quad h \mapsto \begin{cases} \gamma(h), & \text{for } h \neq S_\phi \\ \gamma(a), & \text{else.} \end{cases}$$

It is easy to see that γ' is a morphism. Thus $\chi(H) \leq c$.

Let now be $\gamma(a) \neq \gamma(b)$, for all $\gamma \in \text{Hom}(G, K_c)$. Assume $\chi(H) \leq c$. Then a morphism $\gamma' : H \rightarrow K_c$ exists. Define

$$\gamma^* : VG \rightarrow VK_c, \quad g \mapsto \begin{cases} \gamma'(g), & \text{for } g \notin S_\phi \\ \gamma'(S_\phi), & \text{else.} \end{cases}$$

Again, it is easy to see that a morphism was defined. Therefore $c < \chi(H)$. $\square$

Remark 3.14. The previous proposition applies to simple folds. If one restates it and uses $\chi(H) = \chi(G) + 1$ rather than $\chi(G) < \chi(H)$ then that proposition implies [8, corollary 4.6] that states that $\chi(G) \leq \chi(H) \leq \chi(G) + 1$ if there is a simple fold $\phi \in \text{Hom}(G, H)$.

Example 3.15. Consider the graph G with $VG = \{a, b, c, d, e, f\}$ and $EG = \{ab, bc, cd, de, ef, fa, ac, ae, bf\}$ and the graph H with $S = \{c, e\}$, $VH = \{a, b, f, S, d\}$ and $EH = \{ab, bS, bf, Sd, Sf, Sa, af\}$. Consider furthermore the simple fold $\phi : G \rightarrow H$, with semi-hinge S . Then it is easy to see that $\chi(G) = 3$. It is furthermore easy to see that no coloring of G exists that uses 3 colors only and colors c and e differently. Proposition 3.13 thus implies that $\chi(H) = 4$ which can easily be verified.

The simple proofs of the following two assertions are omitted.

Corollary 3.16. (*Knauer, [20]*)

- (i) *Each color critical graph G is unretractible.*⁴
- (ii) *Odd wheels are unretractible.*

Remark 3.17. There are unretractible graphs that are not color critical.

Proof. The Petersen graph P is known (see [8, p. 122]) to be the Kneser graph $K(2, 5)$. In [8, Proposition 3.13] it was shown that all Kneser graphs are unretractible. So P is unretractible. It is obviously not color critical. $\square$

We can shed more light on unretractivity of graphs by showing how to check that refinements of internal congruences are internal. Compare Figure 1 for the basic idea of the succeeding theorem. For sake of abbreviation we introduce a little further notation. For a graph G the set of its internal congruences and its set of induced subgraphs is denoted by $\text{Int}(G)$ and $\text{Ind}(G)$ respectively. For a set S the set of its partitions is denoted by $P(S)$.

Theorem 3.18. *A connected graph G is unretractible iff it is dunking-free and the following assertions hold:*

- (i) *For all $G' \in \text{Ind}(G)$, $c' \in \text{Int}(G)$, isomorphisms $\alpha' \in \text{Hom}(G', G/c')$, $\Gamma \in VG/c'$ and $\{\Gamma_a, \Gamma_b\} \in P(\Gamma)$ there exists $K \in \text{Ind}(G)$, a glue $\phi \in \text{Hom}(K, G')$ with $S_\phi = \{a, b\}$ and $\Gamma = \phi(S_\phi)$; and*
- (ii) *$xh \in EK$ iff $\gamma_x \mu \in EG$, for all $x \in S_\phi$ and $h \in \phi^{-1} \alpha'^{-1}(VG/c' - \Gamma)$ and for vertices $\gamma_x \in \Gamma_x$ and $\mu \in \alpha' \phi(h)$.*

⁴Note that this assertion already follows from a remark in [10, p. 299].

$$\begin{array}{ccc}
K & \xrightarrow{\alpha''} & G/c'' \\
\phi \downarrow & & \downarrow \pi \\
G' & \xrightarrow{\alpha'} & G/c'
\end{array}$$

FIGURE 1. Commutative diagram illustrating Theorem 3.18.

Proof. Necessity. Let G be unretractable. Then G is dunking-free, because of Proposition 10. That proposition also implies that any refinement of a $c' \in \text{Int}(G)$ is internal. We show that this implies (i) and (ii). Consider $G' \in \text{Ind}(G)$, $c' \in \text{Int}(G)$ and an isomorphism $\alpha' \in \text{Hom}(G', G/c')$. Let $\Gamma \in VG/c'$ be partitioned into Γ_a, Γ_b . Then $c'' = (c' \setminus \Gamma \times \Gamma) \cup (\Gamma_a \times \Gamma_a) \cup (\Gamma_b \times \Gamma_b)$ refines c' and so is internal. Therefore $K \in \text{Ind}(G)$ and an isomorphism $\alpha'' \in \text{Hom}(K, G/c'')$ exist. Let $\pi : VG/c'' \rightarrow VG/c'$ be the canonical mapping and define $\phi = \alpha'^{-1}\pi\alpha''$ then, clearly, ϕ is a glue. Let it be such that $\alpha''(x) = \Gamma_x$, for $x \in S_\phi$. For $x \in S_\phi$ and $h \in \phi^{-1}\alpha'^{-1}(VG/c' - \Gamma) = \alpha''^{-1}(VG/c' - \Gamma)$ holds $xh \in EK$ iff $\Gamma_x\alpha''(h) \in EG/c''$. The latter is equivalent to $\Gamma_x\pi\alpha''(h) \in EG/c'$. Since $\phi = \alpha'^{-1}\pi\alpha''$ we have $\pi\alpha'' = \alpha'\phi$ and that implies (i) and (ii).

Sufficiency. Let G be dunking-free and (i) and (ii) be true. We show that refinements of internal congruences are internal. The unretractivity of G follows then with Proposition 10. We can focus on the case of refinements of one class Γ into two classes Γ_a, Γ_b . Repetition of our argument for other classes than Γ or for refining it into more than two classes implies then the general assertion. Let thus be $c' \in \text{Int}(G)$. Let furthermore be $\Gamma \in VG/c'$ and consider a decomposition of Γ into Γ_a and Γ_b . Let $\alpha' \in \text{Hom}(G', G/c')$ be an isomorphism. Then according to (i) $K \in \text{Ind}(G)$ and a glue $\phi \in \text{Hom}(K, G')$ exist such that $S_\phi = \{a, b\}$ and $\Gamma = \alpha'\phi(S_\phi)$. Assertion (ii) implies that for $x \in S_\phi$, $h \in \phi^{-1}\alpha'^{-1}(VG/c' - \Gamma)$ holds $xh \in EK$ iff $\gamma_x\mu \in EG$, for vertices $\gamma_x \in \Gamma_x$ and $\mu \in \alpha'\phi(h)$. We show that K is isomorphic to G/c'' . Define

$$\alpha'' : VK \rightarrow VG/c'', \quad x \mapsto \begin{cases} \Gamma_x, & \text{for } x \in S_\phi \\ \alpha'\phi(x), & \text{else.} \end{cases}$$

Then α'' is obviously a bijection. For showing that α'' is a morphism we only need to show that $\alpha''(a)\alpha''(h) \in EG/c''$, for $h \in VK \setminus S_\phi$. Let thus be $h \in VK \setminus S_\phi$ and $ah \in EK$. Then because of (ii) we have $\gamma_a\mu \in EG$ for vertices $\gamma_a \in \Gamma_a$ and $\mu \in \alpha'\phi(h)$ and α'' is a morphism. For showing that $(\alpha'')^{-1}$ is a morphism we likewise only need to show that $\Gamma_a\alpha'\phi(h) \in EG/c''$, for $h \in VK \setminus \{a, b\}$, implies $ah \in EK$. But $\Gamma_a\alpha'\phi(h) \in EG/c'$ implies the existence of $\gamma_a \in \Gamma_a$ and $\mu \in \alpha'\phi(h)$ such that $\gamma_a\mu \in EG$. That, however, according to (ii) implies $ah \in EK$. $\square$

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4. OUTLOOK

It would be interesting to look into (1) conclusions regarding the unretractivity of specific graphs, as implied the theorem; (2) characterizing critical graphs in terms of their congruences; and (3) characterizing those graphs for which each internal

congruences is a retractance. Finally, a Galois correspondence between the congruences of a graph and its endomorphisms is induced by the relation that associates to each congruence the endomorphisms whose kernel is that congruence. That Galois correspondence might help better understanding the structure of endomorphism monoids of graphs.

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