

ON SUBPULLBACK AND PULLBACK FLAT, AND SUBFLAT S-POSETS

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For Mati Kilp and Ulrich Knauer in honor of their 65th birthdays

ABSTRACT. In this paper, we show, among other results, that the concept of subpullback and subequalizer flatness for S -posets (*i.e.*, the “ordered” version of the strong flatness for S -acts presented, for instance, in the monograph *Monoids, Acts and Categories* by M. Kilp et al. (New York: Walter de Gruyter, 2000)) is the same as the concept of subpullback and pullback flatness (or the concept of subpullback and equalizer flatness), establish subcompleteness of categories of S -posets, and supplement the characterization of subpullback and subequalizer flat S -posets given in the paper “Lazard’s Theorem for S -posets” by Bulman-Fleming and Laan, *Math. Nachr.* **278**, 1743-1755 (2005).

1. INTRODUCTION

In the sixties, Govorov and Lazard independently established that an R -module P is flat iff it is a filtered (directed) colimit of finitely generated free modules. So, if we define a module P to be L -flat if it is a filtered colimit of finitely generated free modules, then Govorov-Lazard’s theorem simply says that an R -module is flat iff it is L -flat. This result is remarkable in the sense that it provides a “structural” characterization of flat modules (objects) by means of the limiting constructions. It has also generated a natural and sustained interest in establishing the analogues of Govorov-Lazard’s theorem in different universal algebra and functor settings (see, for example, [9] and the references in it).

On the other hand, in the last two decades, there appeared quite interesting publications considering the constructions and results, originated for S -acts (see, *e.g.*, [11]), for S -posets — partially ordered sets with actions of elements of a partially ordered monoid S on them (one may consult [7], [4], [14], [15], and [5] and the references in them for activity in this area). In particular, the tensor product bifunctor and different concepts of (sub)flatness were considered in [4], and as well, an analogue was obtained of the Govorov-Lazard theorem in the context of S -posets. It was shown in [3] (see, also, [8, Corollary 3.7]) that in the Govorov-Lazard theorem in the S -acts context, the concepts of strong flatness and pullback flatness are the same. However, in the Govorov-Lazard theorem in the context of S -posets obtained in [4, Theorem 5.1], the corresponding variation of this fact was not mentioned, and then it was stated by Syd Bulman-Fleming in his talk at the Tartu’07 meeting as an open problem. The present author has recently attempted to prove that for S -posets the subpullback flatness always implies the subequalizer flatness, but it happened

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that the “proof” contained one inaccuracy. Meantime Bulman-Fleming [6] has just recently succeeded in finding a counterexample providing a negative answer to this conjecture. However, all those considerations have brought up a quite interesting spectrum of problems and new promising perspectives concerning the relationship between the (left) ‘subexactness’ and ‘exactness’ of the tensor product bifunctor defined on categories of S -posets. And the present paper addresses some of the problems in this direction.

For the reader’s convenience, all subsequently necessary notions and facts on S -posets are included in Section 2. Then, among other results in Section 3, we obtain one of the main results of the paper, Theorem 3.9, showing that the concept of subpullback and subequalizer flatness for S -posets (*i.e.*, the “ordered” version of the strong flatness for S -acts [11]) is the same as the concept of subpullback and pullback flatness, or the concept of subpullback and equalizer flatness. In Section 4, after introducing the notion of subcompleteness for categories of S -posets, we establish two other central results of the paper — the subcompleteness of categories of S -posets (Theorem 4.2), and an S -poset version (Theorem 4.3) of Theorem 3.6 of [8], supplementing the characterization of subpullback and subequalizer flat S -posets of Theorem 5.1 of [4].

Finally, all notions and facts of categorical algebra, used here without any comments, can be found in [12].

2. PRELIMINARIES

A *partially ordered monoid*, or *pomonoid*, is a monoid S together with a partial order $\leq$ on S such that if $s, t, u \in S$ and $s \leq t$, then $su \leq tu$ and $us \leq ut$. Recall [4] that a *right S -poset* A_S over a pomonoid S is a set A equipped with a partial order $\leq$ and a scalar multiplication $A \times S \rightarrow A$, $(a, s) \mapsto as$, which satisfy the following conditions for all $a, a' \in A$ and $s, t \in S$:

- (1) $a(st) = (as)t$;
- (2) $a1 = a$;
- (3) $a \leq a' \implies as \leq a's$;
- (4) $s \leq t \implies as \leq at$.

Left S -posets over a pomonoid S are defined similarly; and, naturally, morphisms between S -posets are order-preserving homomorphisms of S -acts [11].

From now on, let $\mathcal{P}$ be the category of partially ordered sets, and $\mathcal{P}_S$ and ${}_S\mathcal{P}$ denote the categories of right and left S -posets, respectively, over a pomonoid S . Then (see, for example, [4]; also [14] and [7]), a tensor product bifunctor $- \otimes - : \mathcal{P}_S \times {}_S\mathcal{P} \rightarrow \mathcal{P}$ has been constructed in such a way that it becomes a left adjoint of the *Hom* functor; more precisely, for any right S -poset $A_S \in |\mathcal{P}_S|$ there exists a natural adjunction $A_S \otimes - : {}_S\mathcal{P} \rightleftarrows \mathcal{P} : {}_S\mathcal{P}(A, -)$. Thus, the functor $A_S \otimes -$ is always right exact, *i.e.*, preserves colimits [12, Section V.5]. However, just as for R -modules, the tensor multiplication $A_S \otimes -$ is not always left exact. In the case of modules, a module A is called flat when $A \otimes -$ is left exact, *i.e.*, preserves short exact sequences, which is equivalent to preserving monomorphisms (embeddings), or finite limits. In a non-additive setting like the present one, all those cases are quite different and lead to different, interesting and useful notions of flatness contingent upon the types of finite limits that the corresponding tensor multiplication supposedly preserves (*cf.*, for example, [8]).

It is easy to see that, as for S -acts, coproducts of S -posets are disjoint unions, with S -action and order componentwise defined; and in the category $\mathcal{P}_S$, a free (right) S -poset $\coprod_{i \in I} S_i, S_i \cong S_S, i \in I$, with a basis set I is a coproduct of $|I|$ copies of S_S . Of course, the notion of a free S -poset in the category ${}_S\mathcal{P}$ of left S -posets is the obvious modification of the notion of a free right S -poset. As for S -acts, using the obvious mapping $a \otimes s \mapsto as, a \in A_S \in |\mathcal{P}_S|, s \in {}_SS \in |{}_S\mathcal{P}|$, and observation that a pomonoid S in the common sense (*cf.*, for instance, [10, Section 3]) can be considered as a bi- S -poset ${}_SS$ — an S - S -poset which is both a left and a right S -poset, one can readily verify (see also [7]) that $A_S \otimes {}_SS$ can be naturally considered as a right S -poset, and $A_S \otimes {}_SS$ and A_S are naturally isomorphic in $\mathcal{P}_S$. Thus, applying [12, Section V.5], we have the following observation.

Proposition 2.1. *For any free left S -poset $\coprod_{i \in I} S_i, S_i \cong {}_SS, i \in I$, and any right S -poset $A_S \in |\mathcal{P}_S|$, there is a natural isomorphism $A_S \otimes \coprod_{i \in I} S_i \cong \coprod_{i \in I} A_i, A_i \cong A_S, i \in I$, in the category $\mathcal{P}_S$. In other words, the endofunctors $- \otimes \coprod_{i \in I} S_i$ and $\coprod_{i \in I} (-)_i$ with $(X)_i = X$ for any $X_S \in |\mathcal{P}_S|, i \in I$, on the category $\mathcal{P}_S$ are naturally isomorphic.*

Following [14], we say that an S -subposet ${}_SB$ of a left S -poset ${}_SA \in |{}_S\mathcal{P}|$ is *strongly convex* if for any $a \in A$ and any $b \in B, a \leq b$ implies $a \in B$. An S -subposet ${}_SA \in |{}_S\mathcal{P}|$ is called *indecomposable* if it can not be presented as a coproduct of its proper strongly convex S -subposets. Two elements $x, y \in {}_SA$ are *comparable*, in notations — $x \nparallel y$, if at least one of the inequalities $x \leq y$ or $y \leq x$ is true. In the sequel, the following facts will prove to be useful.

Theorem 2.2. ([14, Theorem 2.3]). *In category ${}_S\mathcal{P}$, every S -poset ${}_SA \in |{}_S\mathcal{P}|$ can be uniquely presented as a coproduct of its strongly convex indecomposable S -subposets—indecomposable components.*

Proposition 2.3. ([14, Proposition 2.6]). *Elements $a, b \in {}_SA \in |{}_S\mathcal{P}|$ are in the same strongly convex indecomposable component of ${}_SA$ iff there are $a_1, \dots, a_n \in A, s_1, t_1, \dots, s_n, t_n \in S$ such that $a \nparallel s_1 a_1, t_1 a_1 \nparallel s_2 a_2, t_2 a_2 \nparallel s_3 a_3, \dots, t_{n-1} a_{n-1} \nparallel s_n a_n, t_n a_n \nparallel b$.*

Recall [4, Definition 3.1] properties (P) and (E) in the context of S -posets:

(P) $A_S \in |\mathcal{P}_S|$ has *property (P)* iff $\forall a, a' \in A \forall s, s' \in S (as \leq a's' \implies \exists a'' \in A \exists u, u' \in S (a''u = a \& a''u' = a' \& us \leq u's'))$;

(E) $A_S \in |\mathcal{P}_S|$ has *property (E)* iff $\forall a \in A \forall s, s' \in S (as \leq as' \implies \exists a' \in A \exists u \in S (a'u = a \& us \leq us'))$.

The categories $\mathcal{P}, \mathcal{P}_S$ and ${}_S\mathcal{P}$ are clearly poset-enriched categories with the pointwise defined order relation on Hom -sets (*i.e.*, $\varphi \leq \psi$ for $\varphi, \psi : A \longrightarrow B$ if and only if $\varphi(a) \leq \psi(a)$ for every $a \in A$). This observation allows us to consider the important concepts of subpullback and subequalizer diagrams for these categories introduced in [4].

Thus, a diagram

$$\begin{array}{ccc} P & \xrightarrow{\pi_2} & N \\ \pi_1 \downarrow & & \downarrow \psi \\ M & \xrightarrow[\varphi]{} & Q \end{array}$$

is said to be a *subpullback diagram* for φ and ψ if

- (1) $\varphi\pi_1 \leq \psi\pi_2$, and
- (2) for any diagram

$$\begin{array}{ccc} R & \xrightarrow{\psi'} & N \\ \varphi' \downarrow & & \downarrow \psi \\ M & \xrightarrow[\varphi]{} & Q \end{array}$$

with $\varphi\varphi' \leq \psi\psi'$, there exists a unique morphism $\eta : R \longrightarrow P$ such that $\pi_1\eta = \varphi'$ and $\pi_2\eta = \psi'$.

As was mentioned in [4], it is clear that for any φ and ψ with the same target in the categories $\mathcal{P}$, $\mathcal{P}_S$ and ${}_S\mathcal{P}$ with the “empty” object, there exists a subpullback diagram in which the object P can actually be presented as

$$P_{\varphi,\psi} = \{(m, n) \mid (m, n) \in M \times N \text{ \& } \varphi(m) \leq \psi(n)\},$$

and π_1 and π_2 are the natural projections. From this one may easily observe the following fact.

Lemma 2.4. *For any $\varphi : M \longrightarrow Q$ and $\psi : N \longrightarrow Q$ in the pullback diagram*

$$\begin{array}{ccc} P & \xrightarrow{\psi'} & N \\ \varphi' \downarrow & & \downarrow \psi \\ M & \xrightarrow[\varphi]{} & Q \end{array},$$

the S -poset P can be considered as the intersection $P_{\varphi,\psi} \cap (P_{\psi,\varphi})^{-1}$ of the S -subposets $P_{\varphi,\psi}$ and $(P_{\psi,\varphi})^{-1}$ of the S -poset $M \times N$, where $P_{\varphi,\psi}$ and $P_{\psi,\varphi}$ are taken from the corresponding subpullback diagrams for φ and ψ , and ψ and φ , respectively, and $(P_{\psi,\varphi})^{-1}$ is the obviously defined converse binary relation of the relation $P_{\psi,\varphi}$.

In a similar fashion, one can define a *subequalizer diagram* $E \longrightarrow M \xrightarrow[\varphi]{\psi} N$ for φ and ψ , and present the object E as $E_{\varphi,\psi} = \{m \mid m \in M \text{ \& } \varphi(m) \leq \psi(m)\}$.

Following [4], an S -poset $A_S \in |\mathcal{P}_S|$ is called *subpullback flat*, or *subequalizer flat*, if the functor $A_S \otimes - : {}_S\mathcal{P} \longrightarrow \mathcal{P}$ preserves subpullback, or subequalizer, diagrams, respectively.

3. ON SUBPULLBACK AND PULLBACK FLAT S -POSETS

From now on, let $\mathbf{1}$ (sometimes with the subscript ‘ S ’ on the right, or on the left) be a terminal object, *i.e.*, a singleton, of the categories $\mathcal{P}$, $\mathcal{P}_S$ and ${}_S\mathcal{P}$. Then, considering for any two S -posets $A_S, B_S \in |\mathcal{P}_S|$ and the only morphisms $\alpha : A_S \longrightarrow \mathbf{1}_S$ and $\beta : B_S \longrightarrow \mathbf{1}_S$ from A_S and B_S to $\mathbf{1}_S$, respectively, a subpullback diagram for α and β

$$\begin{array}{ccc} P & \xrightarrow{\pi_B} & B \\ \pi_A \downarrow & & \downarrow \beta \\ A & \xrightarrow[\alpha]{} & \mathbf{1} \end{array},$$

where $P = P_{\alpha,\beta} = \{(a, b) \mid (a, b) \in A \times B \text{ \& } \alpha(a) = \beta(b) = \mathbf{1}\} = \{(a, b) \mid (a, b) \in A \times B\}$, and π_A and π_B are the natural projections, one immediately comes up with the following observation.

Lemma 3.1. *In $\mathcal{P}_S$, the direct product $A \times B = P_{\alpha,\beta} = P_{\beta,\alpha}$ for $\alpha : A_S \longrightarrow \mathbf{1}_S$ and $\beta : B_S \longrightarrow \mathbf{1}_S$.*

Therefore, for any diagram $A \xrightarrow{f} B$ in $\mathcal{P}_S$, there exist morphisms $(1_A, f)$ and $(1_A, g)$ from A to $A \times B = P_{\alpha,\beta} = P_{\beta,\alpha} = P$ such that $\pi_A(1_A, f) = 1_A = \pi_A(1_A, g)$, $\pi_B(1_A, f) = f$, and $\pi_B(1_A, g) = g$. And let

$$\begin{array}{ccc} P_{(1_A, g), (1_A, f)} & \xrightarrow{p_f} & A \\ p_g \downarrow & & \downarrow (1_A, f) \\ A & \xrightarrow{(1_A, g)} & A \times B = P \end{array} \quad (1)$$

be a subpullback diagram for $(1_A, g)$ and $(1_A, f)$. In $\mathcal{P}_S$, we also have two embeddings $(p_g, p_f) : P_{(1_A, g), (1_A, f)} \longrightarrow A \times A$ and $(1_A, 1_A) : A \longrightarrow A \times A$ to the product $A \times A$ equipped with two natural projections $\pi_1 : A \times A \longrightarrow A$ and $\pi_2 : A \times A \longrightarrow A$, and the following pullback diagram

$$\begin{array}{ccc} E & \xrightarrow{e_1} & A \\ e_2 \downarrow & & \downarrow (1_A, 1_A) \\ P_{(1_A, g), (1_A, f)} & \xrightarrow{(p_g, p_f)} & A \times A \end{array} \quad (2).$$

Proposition 3.2. *The diagram $E \xrightarrow{e} A \xrightarrow[g]{f} B$, where $e = p_f e_2 = p_g e_2$, is a subequalizer diagram for g and f .*

Proof. First, $p_g e_2 = \pi_1(p_g, p_f)e_2 = \pi_1(1_A, 1_A)e_1 = 1_A e_1 = e_1 = 1_A e_1 = \pi_2(1_A, 1_A)e_1 = \pi_2(p_g, p_f)e_2 = p_f e_2 \stackrel{def}{=} e$. Then, one can easily see that $fe = \pi_B(1_A, f)p_f e_2 \geq \pi_B(1_A, g)p_g e_2 = ge$. Now consider a diagram $C \xrightarrow{k} A \xrightarrow[g]{f} B$ in $\mathcal{P}_S$ such that $fk \geq gk$. Then, one may see that $(1_A, f)k \geq (1_A, g)k$, and from (1) we have that there exists a unique morphism $\mu : C \longrightarrow P_{(1_A, g), (1_A, f)}$ such that $p_g \mu = k = p_f \mu$. Therefore, $(p_g, p_f)\mu = (k, k)$ and by (2), there exists a unique morphism $\eta : C \longrightarrow E$ such that $\mu = e_2 \eta$ and $k = e_1 \eta$, and hence $k = p_g \mu = p_g e_2 \eta = e \eta$. Finally, one can routinely justify the uniqueness of η . $\square$

Syd Bulman-Fleming [6] has quite recently showed that the S -poset ${}_S \mathbf{1} \in |{}_S \mathcal{P}|$ in the category ${}_S \mathcal{P}$ over the additive pomonoid of non-positive integers S , under the usual ordering, is subpullback flat but not subequalizer flat. And the main reason for that, as we will see in the sequel, is that the functor $-\otimes {}_S \mathbf{1} : \mathcal{P}_S \longrightarrow \mathcal{P}$ does not preserve pullbacks.

Example 3.3. Let $(1_S, 1_S) : S \longrightarrow S \times S$ be the diagonal S -poset embedding, and $(-1, 0) : S \longrightarrow S \times S$ the S -poset embedding defined by $S \ni 0 \longmapsto (-1, 0) \in S \times S$. Then, one can easily see that

$$\begin{array}{ccc} \emptyset & \longrightarrow & S \\ \downarrow & & \downarrow (-1, 0) \\ S & \xrightarrow{(1_S, 1_S)} & S \times S \end{array}$$

is the pullback of the embeddings $(1_S, 1_S)$ and $(-1, 0)$ in the category $\mathcal{P}_S$. However, using Lemma 3.1, the subpullback flatness of ${}_S \mathbf{1} \in |{}_S \mathcal{P}|$ and Proposition 2.1, we

obtain that

$$\begin{array}{ccc} \mathbf{1} & \longrightarrow & \mathbf{1} \\ \downarrow & & \downarrow (-1, 0) \otimes \mathbf{1}_1 \\ \mathbf{1} & \xrightarrow{(1_S, 1_S) \otimes \mathbf{1}_1} & (S \times S) \otimes_S \mathbf{1} \simeq \mathbf{1} \times \mathbf{1} = \mathbf{1} \end{array}$$

is the pullback of $(1_S, 1_S) \otimes \mathbf{1}_1$ and $(-1, 0) \otimes \mathbf{1}_1$ in the category $\mathcal{P}$.

As has been shown in Proposition 3.2, in the construction of the subequalizer diagram $E \xrightarrow{e} A \xrightleftharpoons[g]{f} B$ for g and f in $\mathcal{P}_S$, pullbacks play a quite active role. In light of this observation and Example 3.3, the following fact is interesting.

Proposition 3.4. *If S -poset ${}_S M \in |\mathcal{SP}|$ ($M_S \in |\mathcal{P}_S|$) is both subpullback and pullback flat, then ${}_S M \in |\mathcal{SP}|$ ($M_S \in |\mathcal{P}_S|$) is also subequalizer flat.*

In order to prove Proposition 3.4, we need first to establish the following fact.

Proposition 3.5. *If $A_S \otimes_S \mathbf{1} = \mathbf{1}$ ($\mathbf{1}_S \otimes_S A = \mathbf{1}$) for a subpullback and pullback flat S -poset $A_S \in |\mathcal{P}_S|$ ($A_S \in |\mathcal{P}_S|$), then A_S (${}_S A$) is also subequalizer flat.*

Proof. By [4, Lemma 3.2] A_S has property **(P)**. Therefore, by [4, Theorem 5.1], we need only to show that A_S has property **(E)**, too. Let $as \leq as'$ for $a \in A$ and $s, s' \in S$. Hence, by property **(P)**, there exist $a' \in A$ and $u, v \in S$ such that $a'u = a = a'v$ and $us \leq vs'$. Then, consider the equalizer diagram ${}_S E \xrightarrow{e} {}_S S \xrightleftharpoons[\tau_v]{\tau_u} {}_S S$, where τ_u and τ_v are right translations, and ${}_S E = \{t \in S \mid tu = tv\}$ in ${}_S \mathcal{P}$. As the functor $A_S \otimes - : {}_S \mathcal{P} \rightarrow \mathcal{P}$ preserves pullbacks and the terminal object ${}_S \mathbf{1}$, from [2, Proposition 2.8.2] we have that it also preserves all finite limits. In particular, $A_S \otimes {}_S E \xrightarrow{1_A \otimes e} A_S \otimes {}_S S \xrightleftharpoons[1_A \otimes \tau_v]{1_A \otimes \tau_u} A_S \otimes {}_S S$ is an equalizer diagram in $\mathcal{P}$.

From the latter and the equation $a'u = a = a'v$, it is easy to see that ${}_S E \neq \emptyset$, and there exist $a'' \in A$ and $t \in {}_S E \subseteq S$ such that $a''t = a'$ and $tu = tv$, and therefore, $a''(tu) = a$ and $(tu)s \leq (tv)s' = (tu)s'$. $\square$

Proof of Proposition 3.4. By Theorem 2.2, an S -poset ${}_S M$ is a coproduct of its strongly convex indecomposable components M_i , $i \in I$, i.e., $M = \coprod_{i \in I} M_i$. In the categories $\mathcal{P}$, $\mathcal{P}_S$ and ${}_S \mathcal{P}$, as one can easily see, subpullbacks, pullbacks, and subequalizers, as well as, by [12, Section V.5], the functor $A_S \otimes -$ commute with coproducts. Therefore, it is enough to prove our statement only for the strongly convex indecomposable S -posets.

Thus, from now on, ${}_S M$ is a subpullback flat, strongly convex indecomposable S -poset. We show that for any elements $a, b \in {}_S M$, we have $\mathbf{1}_S \otimes a = \mathbf{1}_S \otimes b = \mathbf{1}$, i.e., $\mathbf{1}_S \otimes {}_S M = \mathbf{1}$. Indeed, by Proposition 2.3, for any elements $a, b \in {}_S M$, there exist elements $a_1, \dots, a_n \in M$, and $s_1, t_1, \dots, s_n, t_n \in S$ such that $a \nparallel s_1 a_1, t_1 a_1 \nparallel s_2 a_2, t_2 a_2 \nparallel s_3 a_3, \dots, t_{n-1} a_{n-1} \nparallel s_n a_n, t_n a_n \nparallel b$. Then, since by [4, Lemma 3.2] ${}_S M$ has property **(P)**, there exist $c_1, \dots, c_{n+1} \in M$, and $u_1, v_1, \dots, u_{n+1}, v_{n+1} \in S$ such that $a = u_1 c_1, s_1 a_1 = v_1 c_1, t_1 a_1 = u_2 c_2, s_2 a_2 = v_2 c_2, \dots, t_{n-1} a_{n-1} = u_n c_n, s_n a_n = v_n c_n$, and $t_n a_n = u_{n+1} c_{n+1}, b = v_{n+1} c_{n+1}$. Whence we have $\mathbf{1}_S \otimes a = \mathbf{1}_S \otimes u_1 c_1 = \mathbf{1}_S u_1 \otimes c_1 = \mathbf{1}_S \otimes c_1 = \mathbf{1}_S v_1 \otimes c_1 = \mathbf{1}_S \otimes v_1 c_1 = \mathbf{1}_S \otimes s_1 a_1 = \mathbf{1}_S s_1 \otimes a_1 = \mathbf{1}_S \otimes a_1 = \mathbf{1}_S t_1 \otimes a_1 = \mathbf{1}_S \otimes t_1 a_1 = \mathbf{1}_S \otimes u_2 c_2 = \dots = \mathbf{1}_S \otimes b$. Thus, we have shown that $\mathbf{1}_S \otimes {}_S M = \mathbf{1}$, and applying Proposition 3.5, we end the proof. $\square$

From Proposition 3.4 we have

Corollary 3.6. *If S -poset ${}_SM \in |{}_S\mathcal{P}|$ ($M_S \in |\mathcal{P}_S|$) is both subpullback and equalizer flat, then ${}_SM \in |{}_S\mathcal{P}|$ ($M_S \in |\mathcal{P}_S|$) is also subequalizer flat.*

Proof. Indeed, in the category $\mathcal{P}_S$ consider a diagram

$$\begin{array}{ccccc} E & \xrightarrow{e} & P_{\alpha,\beta} & \xrightarrow{\beta'} & B \\ & & \alpha' \downarrow & & \downarrow \beta \\ & & A & \xrightarrow{\alpha} & C \end{array},$$

where the square is a subpullback diagram for α and β , and (E, e) is an equalizer of $\beta\beta'$ and $\alpha\alpha'$. Then, it is easy to see that

$$\begin{array}{ccc} E & \xrightarrow{\beta' e} & B \\ \alpha' e \downarrow & & \downarrow \beta \\ A & \xrightarrow{\alpha} & C \end{array}$$

is a pullback diagram for α and β . Therefore, applying the functor $-\otimes {}_SM : \mathcal{P}_S \rightarrow \mathcal{P}$ to these diagrams, one has that

$$\begin{array}{ccc} E \otimes {}_SM & \xrightarrow{\beta' e \otimes 1_M} & B \otimes {}_SM \\ \alpha' e \otimes 1_M \downarrow & & \downarrow \beta \otimes 1_M \\ A \otimes {}_SM & \xrightarrow{\alpha \otimes 1_M} & C \otimes {}_SM \end{array}$$

is a pullback diagram for $\alpha \otimes 1_M$ and $\beta \otimes 1_M$ in $\mathcal{P}$; hence, $-\otimes {}_SM$ preserves pullbacks, and, using Proposition 3.4, we end the proof. $\square$

Following [2, Section 2.13] (see also [12, Section 9.1]), a *filtered (directed) colimit* is by definition a colimit of a functor $F : I \rightarrow \mathcal{C}$ defined on a filtered (directed) category I . Clearly, the categories $\mathcal{P}$, $\mathcal{P}_S$ and ${}_S\mathcal{P}$, enriched with the “empty” object, are both complete and cocomplete categories, *i.e.*, any functor from a small category to $\mathcal{P}$, $\mathcal{P}_S$ and ${}_S\mathcal{P}$ has both a limit and colimit. As for the filtered colimits in the category **Set** of all sets, an obvious, practicable description of filtered colimits in $\mathcal{P}_S$ is given in [4, Proposition 2.5]. Calling a category J to be *finite* if the set of its morphisms is a finite set, we obtain the main property of filtered colimits in $\mathcal{P}_S$.

Theorem 3.7. *(cf. [2, Theorem 2.13.4], or [12, Theorem 9.2.1]). Let I be a small filtered category, and J a finite category. Then, for any bifunctor $F : I \times J \rightarrow \mathcal{P}_S$ the canonical morphism*

$$\lambda : \text{Colim}_{C \in I} \text{Lim}_{D \in J} F(C, D) \longrightarrow \text{Lim}_{D \in J} \text{Colim}_{C \in I} F(C, D)$$

*is an isomorphism, *i.e.*, in $\mathcal{P}_S$ finite limits commute with filtered colimits:*

$$\text{Colim}_{C \in I} \text{Lim}_{D \in J} F(C, D) \cong \text{Lim}_{D \in J} \text{Colim}_{C \in I} F(C, D).$$

Proof. The canonical morphism λ is easily constructed by using the following two diagrams:

$$\begin{array}{ccc}
F(C, D) & \xleftarrow{\nu_{C,D}} & \text{Lim}_{D \in J} F(C, D) \\
\mu_{C,D} \downarrow & & \downarrow \alpha_C \\
\text{Colim}_{C \in I} F(C, D) & \xleftarrow[\nu_D]{} & \text{Lim}_{D \in J} \text{Colim}_{C \in I} F(C, D)
\end{array}
\quad \text{and}$$

$$\begin{array}{ccc}
\text{Lim}_{D \in J} F(C, D) & \xrightarrow{\mu_D} & \text{Colim}_{C \in I} \text{Lim}_{D \in J} F(C, D) \\
\downarrow \alpha_C & & \downarrow \lambda \\
\text{Lim}_{D \in J} \text{Colim}_{C \in I} F(C, D) & = & \text{Lim}_{D \in J} \text{Colim}_{C \in I} F(C, D)
\end{array},$$

where ν and $\nu_{C,-}$ for each C are limiting cones, and μ , $\mu_{-,D}$ for each D the colimiting cones. Since ν is a cone in C and μ is natural in C , the composite $\mu_{C,D} \nu_{C,D}$ for fixed C is a cone in D ; by the universality of ν , there exist then morphisms α_C for each C making the first squares commute. Since $\mu_{-,D}$ is a cone in C , so is α ; by the universality of μ , there is then a morphism λ making the lower second square commute. It is the desired canonical morphism.

Then, using the construction of filtered colimits in $\mathcal{P}_S$ given in [4, Proposition 2.5] and repeating the arguments in the proof of Theorem 2.13.4 of [2] verbatim, we obtain that λ is an isomorphism. $\square$

Remark 3.8. For a more categorically oriented reader, we want to mention that, in a similar fashion to how it has been done for the category of abelian groups in [2, Proposition 2.13.5 and Corollary 2.13.6], one also can easily justify Theorem 3.7 by taking into consideration the following obvious observation.

Observation: *The forgetful functor $U : \mathcal{P}_S \longrightarrow \mathbf{Set}$ preserves and reflects filtered colimits.*

Using Theorem 3.7, we have the following result.

Theorem 3.9. *For an S -poset ${}_S M \in |\mathcal{S}\mathcal{P}|$, the following assertions are equivalent:*

- (1) ${}_S M$ is both subpullback and pullback flat;
- (2) ${}_S M$ is both subpullback and equalizer flat;
- (3) ${}_S M$ is both subpullback and subequalizer flat.

Proof. The implications (1) $\implies$ (3) and (2) $\implies$ (3) are proved in Proposition 3.4 and Corollary 3.6, respectively.

(1) $\implies$ (2). By Theorem 2.2, an S -poset ${}_S M$ is a coproduct of its strongly convex indecomposable components M_i , $i \in I$, i.e., $M = \coprod_{i \in I} M_i$. As was mentioned earlier, in the categories $\mathcal{P}$, $\mathcal{P}_S$ and $\mathcal{S}\mathcal{P}$ subpullbacks, pullbacks, subequalizers, and equalizers as well as, by [12, Section V.5], the functor $A_S \otimes -$ commute with coproducts. Therefore, it is enough to prove our statement only for the strongly convex indecomposable S -posets. Then, considering ${}_S M$ to be a subpullback flat, strongly convex indecomposable S -poset, from the proof of Proposition 3.4 we have that $\mathbf{1}_S \otimes {}_S M = \mathbf{1}$. Therefore, by [2, Proposition 2.8.2] the functor $- \otimes {}_S M$ preserves all finite limits, in particular, equalizers.

(3) $\implies$ (1). By [4, Theorem 5.1], we can consider ${}_S M$ to be a filtered colimit $\text{Colim}_{i \in I} F_i$ of finitely generated, free left S -posets $F_i, i \in I$, i.e., ${}_S M$ is a colimit of a functor from a small filtered category I to the category $\mathcal{S}\mathcal{P}$ for which, let us say, $i \longmapsto F_i \in |\mathcal{S}\mathcal{P}|$ and $\varphi_{ik} \longmapsto f_{ik} : F_i \longrightarrow F_k$ for any i and $\varphi_{ik} : i \longrightarrow k$ in I

(see [12, p.212]). Let

$$\begin{array}{ccc} P & \xrightarrow{\beta'} & B \\ \alpha' \downarrow & & \downarrow \beta \\ A & \xrightarrow{\alpha} & C \end{array}$$

be a pullback diagram for α and β in the category $\mathcal{P}_S$, and J the finite category $A \xrightarrow{\alpha} C \xleftarrow{\beta} B$. Consider the bifunctor $F : I \times J \rightarrow \mathcal{P}$ defined on the objects by $F(i, A) = A \otimes F_i$, $F(i, B) = B \otimes F_i$, $F(i, C) = C \otimes F_i$, and on morphisms of $I \times J$ by $F(\varphi_{ik}, \alpha) = \alpha \otimes f_{ik} : A \otimes F_i \rightarrow C \otimes F_k$ and $F(\varphi_{ik}, \beta) = \beta \otimes f_{ik} : B \otimes F_i \rightarrow C \otimes F_k$ for any homomorphism $\varphi_{ik} : i \rightarrow k$ in I . Since F_i , $i \in I$, are free S -posets, $-\otimes F_i$ clearly preserves pullbacks, and, hence, $\text{Lim}_J F(i, -) = \text{Lim}_J - \otimes F_i = P \otimes F_i$, i.e.,

$$\begin{array}{ccc} P \otimes F_i & \xrightarrow{\beta' \otimes 1_{F_i}} & B \otimes F_i \\ \alpha' \otimes 1_{F_i} \downarrow & & \downarrow \beta \otimes 1_{F_i} \\ A \otimes F_i & \xrightarrow{\alpha \otimes 1_{F_i}} & C \otimes F_i \end{array}$$

are pullbacks in $\mathcal{P}$ for each $i \in I$. Then, applying Theorem 3.7 one has $\text{Colim}_I P \otimes F_i = \text{Colim}_I \text{Lim}_J F(i, -) \cong \text{Lim}_J \text{Colim}_I F(-, j) = P \otimes \text{Colim}_{i \in I} F_i$, and therefore

$$\begin{array}{ccc} P \otimes \text{Colim}_I F_i & \xrightarrow{\beta' \otimes 1_{\text{Colim}_I F}} & B \otimes \text{Colim}_I F_i \\ \alpha' \otimes 1_{\text{Colim}_I F} \downarrow & & \downarrow \beta \otimes 1_{\text{Colim}_I F} \\ A \otimes \text{Colim}_I F_i & \xrightarrow{\alpha \otimes 1_{\text{Colim}_I F}} & C \otimes \text{Colim}_I F_i \end{array}$$

is a pullback in $\mathcal{P}$, too. $\square$

In light of the results obtained above, one could draw the impression that the subequalizer and pullback flatnesses of an S -poset imply its subpullback flatness. The following example shows that in general it is not the case.

Example 3.10. *Subequalizer + Pullback Flatness $\nRightarrow$ Subpullback Flatness.*

Let a pomonoid S be *discrete*, i.e., the ordering on S is trivial. Then one can easily see that the category $\mathcal{A}_S$ of S -acts is a full subcategory of the category $\mathcal{P}_S$, and filtered colimits of free S -posets are, in fact, S -acts. Therefore, considering, for example, the simplest case when $S = \{1\}$, we have $\mathbf{Set} \subset \mathcal{P}$, and using [4, Theorem 5.1] we have that only sets are subpullback flat posets; however, any poset is subequalizer and pullback (and equalizer) flat.

4. SUBFLAT S -POSETS

4.1. Elements and Ordered Diagrams. For our goals, we have found it reasonable to use slightly modified perception of categorical algebra proposed, for example, in Chapter 1 (in particular, in Sections 1.4 and 1.7) of [1], based on ‘elements’. Namely, for any objects $T, A \in |\mathcal{C}|$ of a category $\mathcal{C}$, any morphism $x : T \rightarrow A$ in $\mathcal{C}$ is regarded as an *element of A defined over T* , or just an *T -element of A* .

Formally, a *graph* I consists of two sets, a set O of *objects* and a set A of *arrows*, and two functions $d^0, d^1 : A \rightarrow O$, called the *source* and the *target*, respectively, of arrows of A . In light of the previous paragraph, any arrow $a : d^0(a) \rightarrow d^1(a)$ of the graph I can be regarded as an element of the target $d^1(a)$ over the source $d^0(a)$, or an *$d^0(a)$ -element of $d^1(a)$* . A *homomorphism* $F : I \rightarrow J$ from a graph I to a

graph J is a function moving objects to objects and arrows to arrows and preserving source and target, *i.e.*, if $f : V_1 \rightarrow V_2$ in I , then $F(f) : F(V_1) \rightarrow F(V_2)$ in J . In the obvious fashion, every category $\mathcal{C}$ produces an underlying graph, $\|\mathcal{C}\|$, in which objects, arrows, source and target maps have the same meaning as in $\mathcal{C}$. Further, if I is a graph, then a *diagram of type I* in a category $\mathcal{C}$ is a graph homomorphism $D : I \rightarrow \|\mathcal{C}\|$. A *finite diagram*, or a *diagram of a finite type I* , is a diagram whose type I has a finite number of objects and arrows.

In this paper, we are mostly dealing with a *finite ordered diagram in a category $\mathcal{C}$* , which is a diagram $D : I \rightarrow \|\mathcal{C}\|$ of a finite type I such that for any arrow $a : d^0(a) \rightarrow d^1(a)$ in I , the set of all $D_a(I)$ -elements of $D(d^1(a))$, where $D_a(I) \stackrel{\text{def}}{=} \{D(d^0(x)) \mid x : d^0(x) \rightarrow d^1(a)\}$, is a poset. In the sequel, however, for the sake of simplicity and without loss of clearness, we will identify arrows and objects of the type I with their D -images, and just consider I itself to be a diagram of the type I in $\mathcal{C}$.

Let I be an ordered diagram in a category $\mathcal{C}$, and $I_0 \stackrel{\text{def}}{=} \{(a_{ik}, a_{jk}) \mid a_{ik} : i \rightarrow k, a_{jk} : j \rightarrow k \text{ in } I \ \& \ (a_{ik} \leq a_{jk})\}$. Then, a *subcone with vertex $V \in |\mathcal{C}|$ over a diagram I* is a family $\{(V, \alpha_i : V \rightarrow i)\}$ of morphisms in $\mathcal{C}$ such that for every $i, j \in I$ there exists a pair $(a_{ik}, a_{jk}) \in I_0$ and $b_{ik}\alpha_i \leq b_{jk}\alpha_j$ for any $(b_{ik}, b_{jk}) \in I_0$. If $\{(W, \beta_i : W \rightarrow i)\}$ is another subcone with vertex W over I , then clearly a *morphism $\gamma : \{(W, \beta_i : W \rightarrow i)\} \rightarrow \{(V, \alpha_i : V \rightarrow i)\}$ between the subcones $\{(W, \beta_i : W \rightarrow i)\}$ and $\{(V, \alpha_i : V \rightarrow i)\}$* is defined to be a morphism $\gamma \in \text{Mor } \mathcal{C}$ such that $\beta_i = \alpha_i \gamma$ for any i . Also it is clear that all subcones over I together with morphisms between them form a category. A terminal object $\{(L, \lambda_i : L \rightarrow i)\}$ of this category, if it exists, is called a *sublimit of the diagram I* . By definition, we obviously have

Proposition 4.1. *Any ordered diagram I up to isomorphism has at most one sublimit.*

A category $\mathcal{C}$ is (*finitely*) *subcomplete* if every ordered (finite) diagram in the category has a sublimit.

4.2. Subcompleteness of Categories of S -posets. Assuming that for any pomonoid S the categories $\mathcal{P}_S$ and ${}_S\mathcal{P}$ are with the added initial, “empty,” object, we obtain the following result.

Theorem 4.2. *For any pomonoid S , the categories $\mathcal{P}_S$ and ${}_S\mathcal{P}$ are subcomplete.*

Proof. Let I be an ordered diagram in $\mathcal{P}_S$, and $I_0 \stackrel{\text{def}}{=} \{(a_{ik}, a_{jk}) \mid a_{ik} : i \rightarrow k, a_{jk} : j \rightarrow k \text{ in } I \ \& \ (a_{ik} \leq a_{jk})\}$. It is obvious that a product of any family of S -posets in $\mathcal{P}_S$ exists and, in fact, is the usual cartesian product of sets with componentwise defined action and order. Therefore, consider the products $\prod_{i \in d^0(I_0)} A_i$ with projections π_i , where $d^0(I_0)$ denotes the set of all sources of components of the pairs in I_0 , and $\prod_{(a_{ik}, a_{jk}) \in I_0} A_{(a_{ik}, a_{jk})}$ with projections $\rho_{(a_{ik}, a_{jk})}$, where $A_{(a_{ik}, a_{jk})} = A_k$ for each pair $(a_{ik}, a_{jk}) \in I_0$. Then, consider two induced morphisms

$$(\times)_{(a_{ik}, a_{jk})} a_{jk} \pi_j : \prod_{i \in d^0(I_0)} A_i \longrightarrow \prod_{(a_{ik}, a_{jk}) \in I_0} A_{(a_{ik}, a_{jk})}$$

and

$$\prod_{(a_{ik}, a_{jk})} (\times) a_{ik} \pi_i : \prod_{i \in d^0(I_0)} A_i \longrightarrow \prod_{(a_{ik}, a_{jk}) \in I_0} A_{(a_{ik}, a_{jk})}.$$

As was mentioned in Section 1, there exists a subequalizer diagram

$$L \xrightarrow{e} \prod_{i \in d^0(I_0)} A_i \xrightarrow[\prod_{(a_{ik}, a_{jk})} (\times) a_{ik} \pi_i]{\prod_{(a_{ik}, a_{jk})} (\times) a_{jk} \pi_j} \prod_{(a_{ik}, a_{jk}) \in I_0} A_{(a_{ik}, a_{jk})}$$

for $\prod_{(a_{ik}, a_{jk})} (\times) a_{ik} \pi_i$ and $\prod_{(a_{ik}, a_{jk})} (\times) a_{jk} \pi_j$. Although tedious, it is routine to verify that $\{(L, \pi_i e : L \longrightarrow A_i)\}$ is a sublimit of the ordered diagram I . $\square$

Adjusting the terminology of [13, Section VII.5, Definition 1.4] and [8] for S -poset settings, we say that an S -poset ${}_S M \in |{}_S \mathcal{P}|$ is *(sub)flat* iff the functor $- \otimes {}_S M : \mathcal{P}_S \longrightarrow \mathcal{P}$ preserves all finite (sub)limits. Then, combining [4, Theorem 5.3] and Theorem 3.9, we obtain the following S -poset version of Theorem 3.6 of [8].

Theorem 4.3. *For an S -poset ${}_S M \in |{}_S \mathcal{P}|$, the following assertions are equivalent:*

- (1) ${}_S M$ has Properties (P) and (E);
- (2) Every surjective S -poset morphism ${}_S N \longrightarrow {}_S M$ is a pure epimorphism;
- (3) Every S -poset morphism ${}_S N \longrightarrow {}_S M$ from a finitely presented S -poset ${}_S N$ factors through a finitely generated free S -poset;
- (4) ${}_S M$ is a directed (filtered) colimit of finitely generated free S -posets;
- (5) ${}_S M$ is subpullback and subequalizer flat;
- (6) ${}_S M$ is subpullback and pullback flat;
- (7) ${}_S M$ is subpullback and equalizer flat;
- (8) ${}_S M$ is a coproduct of subflat S -posets;
- (9) ${}_S M$ is a coproduct of subpullback flat, flat S -posets.

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) and (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) have been established in [4, Theorem 5.1] and Theorem 3.9, respectively.

(5) $\Rightarrow$ (8). By Theorem 2.2, an S -poset ${}_S M$ is a coproduct of its strongly convex, indecomposable components M_i , $i \in I$, i.e., $M = \coprod_{i \in I} M_i$. As was shown in the proof of the implication (1) $\Rightarrow$ (2) of Theorem 3.9, $- \otimes_S M_i$ preserves all finite limits, in particular, all finite products. From the latter and Theorem 4.2 we obtain the statement.

(8) $\Rightarrow$ (9). In particular, ${}_S M$ is a coproduct of subpullback and subequalizer flat S -posets. And therefore, by the equivalences (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7), ${}_S M$ can be presented as a coproduct of strongly convex, indecomposable components, which are simultaneously subpullback and pullback flat S -posets. To finish the proof of the implication, one should only note that, as has been shown in the proof of Proposition 3.4, those components also preserve under tensor multiplication the terminal object $\mathbf{1}_S \in |\mathcal{P}_S|$, and hence, are flat S -posets.

(9) $\Rightarrow$ (7). It is clear since in the category $\mathcal{P}$ equalizers commute with coproducts. $\square$

Remark 4.4. For the reasons mentioned in Example 3.10, one can easily see that, in fact, Theorem 3.6 of [8] for the single-object small categories $\mathbf{C}$ (i.e., $\mathbf{C}$ is a monoid S) is contained in a “discrete” version of Theorem 4.3.

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