

ABSOLUTELY EQUATIONALLY COMPACT MONOIDS

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For Mati Kilp and Ulrich Knauer on the occasion of their 65th birthdays

ABSTRACT. The present survey is devoted to absolutely equationally compact monoids and their generalizations. Examples of monoids from different classes are given.

1. THE DEFINITIONS

We start with the basic definitions of equationally compact algebras.

Definition 1.1. An algebra A is called *equationally compact* if every system of polynomial equations (with constants from A) has a solution in A provided that every finite subsystem has a solution in A .

Equationally compact algebras in a variety $\mathbf{C}$ can be defined more formally in the following way. First, define a free extension of an algebra $A \in \mathbf{C}$ by a set X as an algebra $A[X] \supseteq A$ that contains X as a subset so that for any algebra $B \in \mathbf{C}$, any given homomorphism $f: A \rightarrow B$ and any mapping $h: X \rightarrow B$ uniquely extend to a homomorphism $A[X] \rightarrow B$. An equation over A is represented by an element of $A[X] \times A[X]$. Now we are able to reformulate the definition above: an algebra A is called equationally compact if for any set $S \subseteq A[X] \times A[X]$, there exists a homomorphism $f: A[X] \rightarrow A$ such that $f|_A = id_A$ and $S \subseteq \text{Ker } f$ whenever for an arbitrary finite subset $H \subseteq S$, there exists a homomorphism $f_H: A[X] \rightarrow A$ such that $f_H|_A = id_A$ and $H \subseteq \text{Ker } f_H$.

Definition 1.2. An algebra A is called *weakly equationally compact* if every system of polynomial equations (without constants) has a solution in A provided that every finite subsystem has a solution in A .

For the formal definition of weak equational compactness, denote by $F(X)$ the (absolutely) free algebra generated by the set X . We have that an algebra A is weakly equationally compact if for any set $S \subseteq F[X] \times F[X]$, there exists a homomorphism $f: F[X] \rightarrow A$ with $S \subseteq \text{Ker } f$ whenever for an arbitrary finite subset $H \subseteq S$, there exists a homomorphism $f_H: F[X] \rightarrow A$ such that $H \subseteq \text{Ker } f_H$.

In the following we will consider equational compactness in the category of S -acts, S a monoid. In this case we have the following definitions.

Definition 1.3. A monoid S is called *left absolutely equationally compact* if all left S -acts are equationally compact.

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Definition 1.4. An algebra A is called *f-equationally compact* (1-equationally compact) if every system of polynomial equations with constants from A that contains a finite number of variables (all containing one and the same variable, respectively) has a solution in A provided that every finite subsystem has a solution in A .

Definition 1.5. A monoid S is called *left absolutely f-equationally compact* (left absolutely 1-equationally compact) if all left S -acts are *f-equationally compact* (respectively, 1-equationally compact).

Definition 1.6. A left S -act ${}_S A$ is called *weakly 1-equationally compact* if every finitely solvable system of equations of ${}_S A$ without constants containing only one and the same variable is solvable. A monoid S is called *left absolutely weakly 1-equationally compact* if all left S -acts are weakly 1-equationally compact.

Definition 1.7. A monoid S is called *left absolutely injective* if all left S -acts are injective.

Each polynomial equation in a left S -act A is clearly of one of the types $sx = ty$, $sx = tx$ or $sx = a$ where $s, t \in S$ and $a \in A$. For a system Σ of equations on an S -act define the *graph of Σ* as follows: the set of variables in Σ will be the vertices and there is an edge between vertices x and y if there is an equation of type $sx = ty$ in Σ . The system Σ will be called *connected* if the graph of Σ is connected.

Definition 1.8. We call an S -act ${}_S A$ *c-equationally compact* if it is equationally compact with respect to systems of equations every maximal connected subsystem of which contains constants.

It is proved in [12] that *c-equational compactness* is only a slight generalization of *equational compactness*: every *c-equationally compact* act with zero is in fact *equationally compact*. The major advantage of *c-equational compactness* is that this property carries over to coproducts and vice versa (that is, a coproduct $\coprod A_i$ of S -acts is *c-equationally compact* if and only if every A_i is *c-equationally compact*). That is not true in the case of *equationally compact* S -acts.

The definitions of other notions used in this article can be found in [6].

2. HISTORICAL BACKGROUND - GENERAL ALGEBRAS AND SOME SPECIFIC CLASSES OF ALGEBRAS

The theory of *equationally compact* general algebras was started in the early 1960s by J. Mycielski ([9]) and B. Węglorz ([15]). It was based on ideas that were used primarily by S. Balcerzyk and J. Łoś in studying abelian groups (see, for example, [1] and [7]).

In the 1970s the theory of *equationally compact* algebras was advanced mainly by B. Banaschewski, E. Nelson and W. Taylor. There are two surveys of the results on *equationally compact* algebras, one presented by W. Taylor in [14] and another presented by G. Wenzel in [17].

Recall first that an embedding $A \rightarrow B$ is called *pure* if every finite system of equations with constants in A solvable in B , is solvable in A as well. Among other fundamental results, B. Banaschewski and E. Nelson proved in [3] the following.

Theorem 2.1. *An algebra is equationally compact if and only if it is injective with respect to pure embeddings.*

B. Banaschewski also proved in [2] that an algebra A is equationally compact if and only if any direct limit over A of algebras retractable to A is itself retractable to A .

Next we cite some other earlier results that are most relevant to the topic of this survey.

First, for a monogenic monoid $S = \{1, s, s^2, s^3, \dots, s^n, \dots\}$ generated by an element s , an element a in a left S -act ${}_S A$ is said to have *order* n if 1) $s^m a \neq a$ for all $m \in \mathbb{N}$ and 2) the set $\{b | s^n b = a\}$ contains a minimal element (an element $a \in A$ is called *minimal* if $\{b | sb = a\} = \emptyset$). Denote by C_∞^∞ the following S -act: $\{c_i | i \in \mathbb{Z}, sc_i = c_{i+1}\}$ and by C_n the n -element cycle $\{c_0, \dots, c_{n-1} | sc_i = c_{i+1(\text{mod } n)}\}$. Then we have ([16], Theorem 2) the following.

Theorem 2.2. *The unary algebra $A = (A, s)$ is equationally compact if and only if*

- 1) *for every $a \in A$, if $\lim(n; n = \text{order of } a) = \aleph_0$ then a is of infinite order;*
- 2) *A contains either C_∞^∞ or some $C_n, n \geq 1$.* □

It follows from the last theorem that a finite monogenic monoid is absolutely equationally compact.

B. Banaschewski solved in [4] a number of problems in the category of G -acts, G a group. Later we need the following.

Theorem 2.3. *All G -acts are equationally compact if and only if all subgroups of G are finitely generated.*

As was proved in [12, Proposition 3.3], c -equationally compact S -acts have the following diagrammatic description.

Theorem 2.4. *A left S -act ${}_S A$ is c -equationally compact if and only if for every diagram*

$$\begin{array}{ccc} {}_S X & \xrightarrow{\mu} & {}_S Y \\ \downarrow \varphi & & \\ {}_S A & & \end{array}$$

where μ is a pure monomorphism such that ${}_S Y = \coprod_I Y_i$ implies $\mu({}_S X) \cap {}_S Y_i \neq \emptyset$, for each $i \in I$, there exists a homomorphism $\bar{\varphi}: {}_S Y \rightarrow {}_S A$ such that $\varphi = \bar{\varphi}\mu$.

Article [8] considers equational compactness of projection algebras. A projection algebra is defined as an $\mathbb{N}^\infty$ -act where $\mathbb{N}^\infty$ is the monoid of natural numbers with the binary operation $m \cdot n = \min\{m, n\}$ and the identity element ∞ adjoined. It turned out [8, Theorem 3.1] that *equational compactness* and *injectivity* coincide for projection algebras.

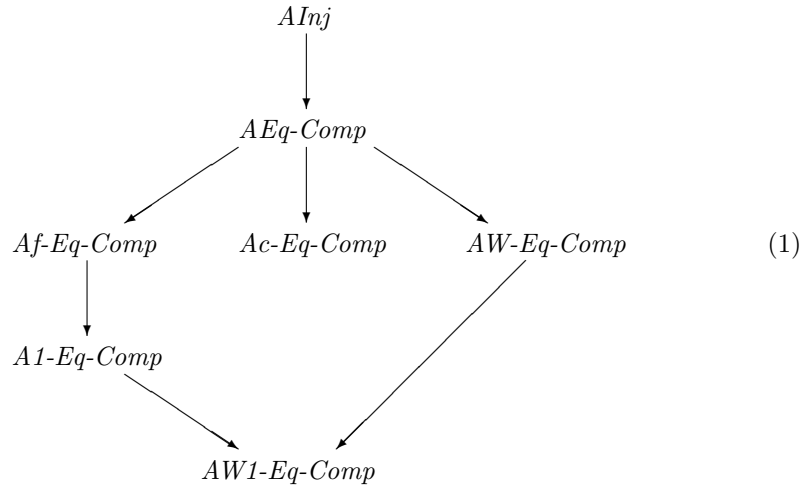
Notice also that the theory of equationally compact S -acts and monoids is not very well developed yet and therefore the handbook [6] does not consider equational compactness.

3. THE CLASS OF ABSOLUTELY EQUATIONALLY COMPACT MONOIDS

Introduce the following notations:

$AINj$ — the class of all left absolutely injective monoids,
 $AEq-Comp$ — the class of all left absolutely equationally compact monoids,
 $Ac-Eq-Comp$ — the class of all left absolutely c -equationally compact monoids,
 $Af-Eq-Comp$ — the class of all left absolutely f -equationally compact monoids,
 $A1-Eq-Comp$ — the class of all left absolutely 1-equationally compact monoids,
 $AW-Eq-Comp$ — the class of all left absolutely weakly equationally compact monoids,
 $AW1-Eq-Comp$ — the class of all left absolutely weakly 1-equationally compact monoids.

From Theorem 2.1 and from the definitions above we have the following diagram of inclusions ($X \rightarrow Y$ means that X is a subclass of Y):



As we will see in this section, the classes $AINj$, $Af-Eq-Comp$, $A1-Eq-Comp$ and $AW1-Eq-Comp$ are fully described. Moreover, all inclusions but one in the last diagram are strong.

For an element $a \in {}_S A$ of a left S -act ${}_S A$ and a congruence ρ on ${}_S A$, denote by ρ^a the following left congruence on S :

$$u\rho^a v \Leftrightarrow uapva.$$

In the most simple case where systems of equations contain one variable only and do not contain constants we have the following ([10], Lemma 6.5):

Theorem 3.1. *A monoid S is left absolutely weakly 1-equationally compact if and only if for every left congruence ρ of S , there exists an element $u \in S$ and a finitely generated subcongruence $\psi \subseteq \rho$ such that $\rho \subseteq \psi^u$.*

The next theorem shows that the classes $Af-Eq-Comp$ and $A1-Eq-Comp$ in fact coincide (cf. [13, Theorem]).

Theorem 3.2. *The following conditions on a monoid S are equivalent:*

- 1) S is left absolutely f -equationally compact;
- 2) S is left absolutely 1-equationally compact;
- 3) All left ideals of S are finitely generated and for every left congruence ρ of S and every finite set $\{s_1, \dots, s_k\} \subseteq S$ there exist an element $u \in S$ and a finitely

generated subcongruence $\psi \subseteq \rho$ such that $(s_i, s_i u) \in \rho$ for every i and $\rho \subseteq \psi^u \cup (\rho^u|_J)$, where J is the left ideal of S generated by the set $\{s_1, \dots, s_k\}$.

We will later make use of the following example.

Example 3.3. Let $S = \{1, s, s^2, s^3, \dots, s^n, \dots\}$ be an infinite monogenic monoid. The system of equations $\{x_1 = sx_2, x_2 = sx_3, x_3 = sx_4, \dots\}$ is finitely solvable but not solvable in S . We have that ${}_S S$ is not weakly equationally compact as a left S -act, and therefore not equationally compact as well.

Now we are able to formulate our main result.

Theorem 3.4. *All inclusions in the diagram (1) are strong except one: $Af-Eq-Comp = A1-Eq-Comp$.*

Proof. We have the following memberships:

- (1) An arbitrary nontrivial finite group G belongs to $AEq-Comp \setminus AInj$. For, $G \in AEq-Comp$ by Theorem 2.3. On the other hand, every left absolutely injective monoid contains zero by [6, Theorem 5.13] and therefore $G \notin AInj$.
- (2) An infinite monogenic monoid S belongs to $Af-Eq-Comp \setminus AEq-Comp$. For, S satisfies the maximum condition on congruences and therefore $S \in Af-Eq-Comp$ by [13, Corollary 1]. On the other hand, it follows from Example 3.3 that $S \notin AEq-Comp$. The same arguments are valid for showing that S belongs to $AW1-Eq-Comp \setminus AWEq-Comp$.
- (3) The multiplicative interval of real numbers $([0,1], \cdot)$ belongs to $AW1-Eq-Comp \setminus Af-Eq-Comp$. For, every monoid that contains zero belongs obviously to $AW1-Eq-Comp$. On the other hand, $([0,1], \cdot)$ is an ideal of $([0,1], \cdot)$ that is not finitely generated and therefore $([0,1], \cdot) \notin Af-Eq-Comp$ by Theorem 3.2. The same arguments are valid for showing that $([0,1], \cdot)$ belongs to $AWEq-Comp \setminus AEq-Comp$.
- (4) A group G that has subgroups which are not finitely generated belongs to $Ac-Eq-Comp \setminus AEq-Comp$. For, $G \in Ac-Eq-Comp$ by Theorem 2.4 (in the case of groups, the acts ${}_S X$ and ${}_S Y$ coincide in the formulation of Theorem 2.4) and $G \notin AEq-Comp$ by Theorem 2.3.

□

4. OPEN PROBLEMS AND DISCUSSIONS

We list first the open problems related to the diagram (1) of the previous section:

Problem 1. Provide descriptions of the following classes of monoids:

- 1) $AEq-Comp$ — the class of all left absolutely equationally compact monoids;
- 2) $Ac-Eq-Comp$ — the class of all left absolutely c -equationally compact monoids;
- 3) $AW-Eq-Comp$ — the class of all left absolutely weakly equationally compact monoids.

Another set of open problems concerns the relation between generalizations of equational compactness as well as between equational compactness and congruence compactness (an algebra is called *congruence compact* if every filter base consisting of classes of congruences has a nonempty intersection). Here only some partial results are obtained so far. For example, it is proved in [11, Proposition 2.12] that every congruence compact act over a commutative monoid is f-equationally

compact. It is also known that every congruence compact act over a monogenic monoid and over a commutative group is equationally compact [11, Proposition 3.7 and Corollary 4.3, respectively]. On the other hand, in the case of general monoids, congruence compact acts need not even be weakly 1-equationally compact by [11, Example 2.11]. Therefore we have, for example, the following unsolved problems.

Problem 2. Provide descriptions of:

- 1) commutative monoids over which all congruence compact acts are equationally compact;
- 2) monoids over which all f -equationally compact acts are equationally compact;
- 3) monoids over which all c -equationally compact acts are equationally compact.

Note here that the question “when all equationally compact acts are congruence compact?” has a trivial answer: never (an S -act consisting of an infinite number of zero elements is equationally compact but not congruence compact).

We get a set of slightly different problems if we consider a monoid S as a left S -act over itself.

Definition 4.1. A monoid S is called left (weakly, f -, 1-, c -, weakly 1-) self-equationally compact if ${}_S S$ is (weakly, f -, 1-, c -, weakly 1-) equationally compact.

Self-equationally compact monoids are described for a few classes of monoids only. For example, by Theorem 2.2 and Example 3.3, a monogenic monoid is self-equationally compact if and only if it is finite. Self-equationally compact groups (as left acts over itself) are described in [4, Proposition 1]. Theorem A of [5] described equationally compact semilattices; it was also proved that a semilattice is equationally compact if and only if it is 1-equationally compact (Corollary of [5]). As for a semilattice S , 1-equational compactness of S as a semilattice is the same as 1-equational compactness of S as an S -act, Theorem A of [5] describes in fact 1-self-equationally compact semilattices (as left acts over itself). Among the type of problem not yet solved we have the following.

Problem 3. Provide descriptions of:

- 1) self-equationally compact monoids;
- 2) 1-self-equationally compact monoids;
- 3) weakly self-equationally compact monoids.

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