

ON COPRODUCTS OF QUANTALE ALGEBRAS

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ABSTRACT. The paper constructs coproducts of objects in the category $Q\text{-}\mathbf{Alg}$ of algebras over a given unital commutative quantale Q . In the construction, to obtain a suitable quotient, we use the technique of nucleus, adjusted for our specific context.

1. INTRODUCTION

While the study of quantale-like structures goes back to the work of Ward and Dilworth [22] on residuated lattices (notwithstanding that the term itself was introduced by Mulvey [13] in order to provide a lattice theoretic setting for studying non-commutative C^* -algebras, as well as a constructive foundations for quantum mechanics), there has recently been much interest in quantales in a variety of contexts. In particular, quantales are extremely useful in theoretical computer science. A class of quantales, called Girard quantales, provides the partially ordered semantics for Girard's linear logic [8]. Quantales have been extensively used to study process semantics as well [1].

This paper is meant as a contribution to the theory of quantales. We begin by introducing the category $Q\text{-}\mathbf{Alg}$ of algebras over a given unital commutative quantale Q (shortly Q -algebras). It generalizes the categories $\mathbf{Quant}$ of quantales [18] and $Q\text{-}\mathbf{Mod}$ of (left) modules over Q [15, 20]. Since the category $Q\text{-}\mathbf{Alg}$ is concrete over the category of semigroups we use the idea of monoid rings of [11] to construct a free Q -algebra from a given semigroup. It follows (see, e.g., [12]) that the category $Q\text{-}\mathbf{Alg}$ is a monadic construct and therefore is cocomplete. We are interested in an explicit construction of coproducts of quantale algebras.

The category in question is a special case of the categories of distributive algebras in linear categories introduced by Banaschewski and Pultr in [5] to study similarities in the construction of coproducts of locales and commutative rings with unit as well as coincidence of these constructions with the natural tensor product of the underlying additive structures (see also [4]). The authors present a method of obtaining colimits in such categories. As an example they consider coproducts of distributive algebras. The crucial point is to get a certain quotient object in the respective category. Some authors use the standard technique of nucleus to construct such quotients in categories of ordered algebraic structures (see, e.g., [17] for the category of locales). Following the example we introduce the notion of quantale algebra nucleus which generalizes the notions of quantic nucleus [14] and module nucleus [16]. With the help of the notion we illustrate the construction of

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[5] in case of the category $Q\text{-}\mathbf{Alg}$ providing an explicit description of the elements of the coproduct in question.

2. THE CATEGORY $Q\text{-}\mathbf{Alg}$

For convenience of the reader in this section we recall from [21] the definition of the category $Q\text{-}\mathbf{Alg}$ of algebras over a given unital commutative quantale Q . Start by recalling the definition of quantale (cf. [18]).

Definition 2.1. A *quantale* is a triple $(Q, \leq, \otimes)$ such that

- (i) $(Q, \leq)$ is a complete lattice;
- (ii) $(Q, \otimes)$ is a semigroup;
- (iii) $q \otimes (\bigvee S) = \bigvee_{s \in S} (q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S} (s \otimes q)$ for every $q \in Q$ and $S \subseteq Q$.

A quantale Q is said to be *unital* provided that there exists an element $1 \in Q$ such that $(Q, \otimes, 1)$ is a monoid. Q is said to be *commutative* provided that $q_1 \otimes q_2 = q_2 \otimes q_1$ for every $q_1, q_2 \in Q$.

Notice that every quantale, being a complete lattice, has the largest element “ $\top$ ” and the smallest element “ $\perp$ ”. The following are typical examples of quantales.

Example 2.2. Every frame, i.e., a complete lattice L with $a \wedge (\bigvee S) = \bigvee_{s \in S} (a \wedge s)$ for every $a \in L$ and $S \subseteq L$ (cf. [10]), is a commutative quantale with $\otimes = \wedge$ and $1 = \top$. In particular, the chain $\mathbf{2} = \{0, 1\}$ is a quantale.

Example 2.3. Let $(A, \cdot)$ be a semigroup. The power-set $\mathcal{P}(A)$ is a quantale where joins are unions and $\otimes$ is given by $S \otimes T = \{s \cdot t \mid s \in S, t \in T\}$. If $(A, \cdot, 1)$ is a monoid, then $\mathcal{P}(A)$ is unital, with $\{1\}$ being the unit. If $(A, \cdot)$ is commutative, then so is $\mathcal{P}(A)$.

Notice that Example 2.3 provides the free quantale over a given semigroup. We will generalize the construction later on while producing free quantale algebras from semigroups.

Example 2.4. Let X be a set and let $\mathcal{R}(X)$ denote the set of all relations on X . $\mathcal{R}(X)$ is a quantale where joins are unions and $\otimes$ is given by $S \otimes T = \{(x, y) \in X \times X \mid \text{there exists } z \in X \text{ with } (x, z) \in T \text{ and } (z, y) \in S\}$. $\mathcal{R}(X)$ is unital, with the diagonal relation Δ being the unit.

It is shown in [6] that every unital quantale is isomorphic to a *relational quantale*, i.e., to a subset of $\mathcal{R}(X)$ containing Δ and closed under composition of relations, with joins being (in general) different from unions.

Definition 2.5. Let Q and S be quantales. A map $Q \xrightarrow{f} S$ is a *quantale homomorphism* provided that f preserves $\otimes$ and arbitrary joins.

Definitions 2.1 and 2.5 give the category **Quant** of quantales and quantale homomorphisms. On the next step we need the category $Q\text{-}\mathbf{Mod}$ of (unital left) modules over a given quantale Q (cf. [15, 19, 20]). The definition is motivated by the category $R\text{-}\mathbf{Mod}$ of (unital left) modules over a ring R (cf. [3, 9]).

Definition 2.6. Let Q be a unital quantale. $Q\text{-}\mathbf{Mod}$ is the category, the objects of which (called *unital left Q -modules*) are pairs $(A, *)$ where A is a complete lattice and $Q \times A \xrightarrow{*} A$ is a map such that:

- (i) $q * (\bigvee S) = \bigvee_{s \in S} (q * s)$ for every $q \in Q$ and $S \subseteq A$;
- (ii) $(\bigvee T) * a = \bigvee_{t \in T} (t * a)$ for every $a \in A$ and $T \subseteq Q$;
- (iii) $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every $q_1, q_2 \in Q$ and $a \in A$;
- (iv) $1 * a = a$ for every $a \in A$.

The morphisms of the category (called *unital left Q -module homomorphisms*) are join-preserving maps $A \xrightarrow{f} B$ with $f(q * a) = q * f(a)$ for every $a \in A$ and $q \in Q$.

For shortness sake from now on “ Q -module” means “unital left Q -module”. Some properties of the category $Q\text{-}\mathbf{Mod}$ are considered in [20], e.g., that it is a monadic construct. In particular, one can easily see that the category $\mathbf{2}\text{-}\mathbf{Mod}$ is isomorphic to the category $\mathbf{CSLat}(\bigvee)$ of complete lattices and join-preserving maps.

On the last step we define the category $Q\text{-}\mathbf{Alg}$ of algebras over a given unital commutative quantale Q . The definition is motivated by the category $K\text{-}\mathbf{Alg}$ of algebras over a given commutative ring K with identity (cf. [3, 9]).

Definition 2.7. Let Q be a unital commutative quantale. $Q\text{-}\mathbf{Alg}$ is the category, the objects of which (called *Q -algebras*) are Q -modules $(A, *)$ such that:

- (i) A is a quantale;
- (ii) $q * (a \otimes b) = (q * a) \otimes b = a \otimes (q * b)$ for every $a, b \in A$ and $q \in Q$.

The morphisms of the category (called *Q -algebra homomorphisms*) are quantale homomorphisms $A \xrightarrow{f} B$ which are also Q -module homomorphisms.

It is easy to see that the category $\mathbf{2}\text{-}\mathbf{Alg}$ is isomorphic to the category $\mathbf{Quant}$ of quantales. For convenience sake from now on we fix a unital commutative quantale Q and consider the category $Q\text{-}\mathbf{Alg}$ of Q -algebras.

3. $Q\text{-}\mathbf{Alg}$ IS A MONADIC CONSTRUCT

This section is devoted to some properties of the category $Q\text{-}\mathbf{Alg}$ needed in the sequel (cf. [21]). Start with the remark that there exists the following commutative triangle of the obvious forgetful functors (notice that $\mathbf{Set}$ is the category of sets and maps and $\mathbf{SGrp}$ is the category of semigroups and semigroup homomorphisms):

$$\begin{array}{ccc}
 Q\text{-}\mathbf{Alg} & \xrightarrow{U} & \mathbf{Set} \\
 & \searrow V & \nearrow W \\
 & \mathbf{SGrp} &
 \end{array}$$

Theorem 3.1. *The functor $Q\text{-}\mathbf{Alg} \xrightarrow{V} \mathbf{SGrp}$ has a left adjoint.*

Proof. Let $(S, \cdot)$ be a semigroup. It is enough to find a free Q -algebra over S . Let Q^S be the set of all maps $S \xrightarrow{\alpha} Q$. Clearly Q^S is a Q -module with the operations lifted point-wise from Q . By analogy with monoid rings of [11] we define a quantale operation on Q^S by: $(\alpha \otimes \beta)(z) = \bigvee_{z=x \cdot y} \alpha(x) \otimes \beta(y)$. Straightforward computations show that $(Q^S, \otimes)$ is a quantale. For every $x \in S$ there exists a map $\alpha_x \in Q^S$ defined by: $\alpha_x(y) = 1$, if $y = x$; otherwise, $\alpha_x(y) = \perp$. The family $(\alpha_x)_{x \in S}$ gives rise to a $\mathbf{SGrp}$ -morphism $S \xrightarrow{\eta_S} VQ^S$ defined by: $\eta_S(x) = \alpha_x$. Straightforward

computations show that (η_S, Q^S) is a V -universal arrow for S , i.e., for every **SGrp**-morphism $S \xrightarrow{f} VA$ there exists a unique Q -**Alg**-morphism $Q^S \xrightarrow{\bar{f}} A$ (given by: $\bar{f}(\alpha) = \bigvee_{x \in S} \alpha(x) * f(x)$) making the triangle

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & VQ^S \\ & \searrow f & \downarrow V\bar{f} \\ & & VA \end{array}$$

commute. $\square$

The following theorem will be heavily used in the sequel. For the sake of convenience we provide its version from [2].

Theorem 3.2. *The forgetful functor $\mathbf{SGrp} \xrightarrow{W} \mathbf{Set}$ has a left adjoint.*

Proof. Let X be a set. It is enough to construct a free semigroup over X . Let X^* be the set of all non-empty words $(x_1, x_2, \dots, x_n)$ formed from elements of X . Define the semigroup operation “ $\cdot$ ” on X^* by concatenation, i.e., $(x_1, x_2, \dots, x_n) \cdot (y_1, y_2, \dots, y_m) = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m)$. The map $X \xrightarrow{\eta_X} WX^*$ defined by: $\eta_X(x) = (x)$ provides a W -universal arrow for X . $\square$

Corollary 3.3. *The forgetful functor $Q\text{-}\mathbf{Alg} \xrightarrow{U} \mathbf{Set}$ has a left adjoint.*

Proof. Use Theorems 3.1 and 3.2 as well as the fact that adjoint situations can be composed. $\square$

Theorem 3.4. *The category $Q\text{-}\mathbf{Alg}$ is a monadic construct.*

Proof. The category $Q\text{-}\mathbf{Alg}$ is an *equationally-definable class* (cf. Definition 1.5.34 of [12]), i.e., its objects can be described by (a proper class of) operations and equations. Now use Corollary 3.3 as well as Chapter 1 of [12]. $\square$

The monad of Theorem 3.4 is induced by the adjoint situation of Corollary 3.3. Moreover, Theorem 3.4 together with Propositions 8.29 and 20.34 of [2] implies the following corollary.

Corollary 3.5. *The category $Q\text{-}\mathbf{Alg}$ is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations. Monomorphisms (extremal epimorphisms) in $Q\text{-}\mathbf{Alg}$ are precisely those morphisms that are injective (surjective) functions.*

We end the section with a simple (but useful) lemma.

Lemma 3.6. *Surjective $Q\text{-}\mathbf{Alg}$ -morphisms are final, i.e., given a commutative triangle*

$$\begin{array}{ccc} UA & & \\ Ue \downarrow & \searrow Uf & \\ UB & \xrightarrow{h} & UC \end{array}$$

with e surjective, it follows that h is a $Q\text{-}\mathbf{Alg}$ -morphism.

Notice that Lemma 3.6 can be readily applied to the categories $\mathbf{CSLat}(\vee)$ and $\mathbf{Quant}$ as well.

4. COLIMITS IN CATEGORIES WITH QUOTIENTS

It was already mentioned in Introduction that the category $Q\text{-}\mathbf{Alg}$ is a special case of the categories of distributive algebras in linear categories [5] (in notations of [5] $Q\text{-}\mathbf{Alg}$ is just the category $\text{Distr}(Q\text{-}\mathbf{Mod}, \mathbb{T})$ with $\mathbb{T}$ being the theory of semi-groups). The authors of [5] show a method of obtaining colimits in such categories. In this section we provide a simplified modification of their construction adjusted for our needs. Start with a preliminary definition.

Definition 4.1. Let $\mathbf{A} \xrightarrow{U} \mathbf{X}$ be a functor. The category $\mathbf{A}$ is said to *have U -quotients* provided that for every sink $\mathcal{S} = (UA_i \xrightarrow{f_i} UA)_{i \in I}$ in $\mathbf{X}$ there exists an $\mathbf{A}$ -morphism $A \xrightarrow{\varphi} B$ (called an *U -quotient w.r.t. $\mathcal{S}$*) such that $UA_i \xrightarrow{U\varphi \circ f_i} UB = UA_i \xrightarrow{U\bar{f}_i} UB$ for every $i \in I$ and, moreover, every $\mathbf{A}$ -morphism with the same property can be factorized uniquely through φ .

The next lemma provides a sufficient condition for having U -quotients.

Lemma 4.2. *Let $\mathbf{A} \xrightarrow{U} \mathbf{X}$ be a functor. If $\mathbf{A}$ is $(\text{Epi}, U\text{-Initial Source})$ -factorizable, then $\mathbf{A}$ has U -quotients.*

Proof. Given a sink $\mathcal{S} = (UA_i \xrightarrow{f_i} UA)_{i \in I}$ in $\mathbf{X}$, let $\mathcal{L} = (A \xrightarrow{\varphi_j} B_j)_{j \in J}$ be the source of all $\mathbf{A}$ -morphisms $A \xrightarrow{\varphi_j} B_j$ with $UA_i \xrightarrow{U\varphi_j \circ f_i} UB_j = UA_i \xrightarrow{U\bar{f}_i^j} UB_j$ for every $i \in I$. Let $\mathcal{L} = \mathcal{M} \circ \varphi$ be an $(\text{Epi}, U\text{-Initial Source})$ -factorization of $\mathcal{L}$. Straightforward computations show that φ is an U -quotient w.r.t. $\mathcal{S}$. $\square$

It appears that for faithful functors the notion of Definition 4.1 is rather fruitful.

Lemma 4.3. *Let $(\mathbf{A}, U)$ be a concrete category over $\mathbf{X}$. Suppose $\mathbf{A}$ has U -quotients and U has a left adjoint. Then U detects colimits, i.e., a functor $\mathbf{I} \xrightarrow{D} \mathbf{A}$ has a colimit whenever UD has one.*

Proof. Let $\mathbf{I} \xrightarrow{D} \mathbf{A}$ be a functor and let $(UD_i \xrightarrow{q_i} X)_{i \in \text{Ob}(\mathbf{I})}$ be a colimit of UD in $\mathbf{X}$. Choose an adjoint situation $(\eta, \epsilon) : F \dashv U : \mathbf{A} \rightarrow \mathbf{X}$ and let $FX \xrightarrow{\varphi} A$ be an U -quotient for the sink $(UD_i \xrightarrow{\eta_X \circ q_i} UFX)_{i \in \text{Ob}(\mathbf{I})}$. Straightforward computations show that the sink $(D_i \xrightarrow{U\varphi \circ \eta_X \circ q_i} A)_{i \in \text{Ob}(\mathbf{I})}$ is a colimit of D in $\mathbf{A}$. $\square$

The following lemma together with Theorem 3.1 shows that the aforesaid procedure is applicable to the concrete category $(Q\text{-}\mathbf{Alg}, V)$ over $\mathbf{SGrp}$.

Lemma 4.4. *The category $Q\text{-}\mathbf{Alg}$ is $(\text{Epi}, V\text{-Initial Source})$ -factorizable.*

Proof. Given a source $\mathcal{L} = (A \xrightarrow{\varphi_i} A_i)_{i \in I}$ in $\mathbf{A}$, let $\sim$ be the congruence on A defined by $a \sim b$ iff $\varphi_i(a) = \varphi_i(b)$ for every $i \in I$. The quotient map $A \xrightarrow{\phi} A/\sim$ together with the obvious mono-source $\mathcal{M}$ provide the required factorization. $\square$

Unfortunately, Lemma 4.4 does not provide an explicit construction of V -quotients. The next remark makes up the deficiency.

Remark 4.5. Let $\mathcal{S} = (VA_i \xrightarrow{f_i} VA)_{i \in I}$ be a sink in $\mathbf{SGrp}$. For every $i \in I$ define a binary relation $R_i \subseteq A_i \times A_i$ as follows (notice that every map $X \xrightarrow{f} Y$ gives rise

to the standard image operator $\mathcal{P}(X) \xrightarrow{f^\rightarrow} \mathcal{P}(Y)$:

$$\left\{ \left(f_i \left(\bigvee T \right), \bigvee f_i^\rightarrow(T) \right) \mid T \subseteq A_i \right\} \cup \{ (f_i(q * a), q * f_i(a)) \mid a \in A_i, q \in Q \}$$

and set $\mathbf{R} = \bigcup_{i \in I} \mathbf{R}_i$. Let $\langle \mathbf{R} \rangle$ be the smallest congruence relation generated by $\mathbf{R}$

and let $A_{\mathbf{R}} = A / \langle \mathbf{R} \rangle$. The quotient map $A \xrightarrow{\phi} A_{\mathbf{R}}$ is a V -quotient w.r.t. $\mathcal{S}$.

In the following two sections we give an explicit description of the quotient map $A \xrightarrow{\phi} A_{\mathbf{R}}$ of Remark 4.5 using the standard technique of nucleus adjusted for our needs.

5. QUANTALE ALGEBRA NUCLEI

In this section we recall from [21] the notion of quantale algebra nucleus. It generalizes the concepts of quantic nucleus (cf. Definition 3.1.1 in [18]) and module nucleus (cf. Definition 2.1 in [16]).

Definition 5.1. Let A be a Q -algebra. A *quantale algebra nucleus* on A is a map $A \xrightarrow{j} A$ such that for every $a, b \in A$ and $q \in Q$:

- (i) $a \leq b$ implies $j(a) \leq j(b)$;
- (ii) $a \leq j(a)$;
- (iii) $(j \circ j)(a) \leq j(a)$;
- (iv) $j(a) \otimes j(b) \leq j(a \otimes b)$;
- (v) $q * j(a) \leq j(q * a)$.

For shortness sake from now on “nucleus” means “quantale algebra nucleus”. The following proposition shows some consequences of Definition 5.1.

Proposition 5.2. Let j be a nucleus on a Q -algebra A . For every $a, b \in A$, $q \in Q$ and $S \subseteq A$:

- (i) $(j \circ j)(a) = j(a)$;
- (ii) $j(\bigvee S) = j(\bigvee j^\rightarrow(S))$;
- (iii) $j(a \otimes b) = j(j(a) \otimes b) = j(a \otimes j(b))$;
- (iv) $j(q * a) = j(q * j(a))$.

Proof. The proof consists of straightforward computations. As an example we show (i) and (iv).

- (i) Given $a \in A$, $j(a) \leq (j \circ j)(a)$ and therefore $(j \circ j)(a) = j(a)$.
- (iv) Given $a \in A$ and $q \in Q$, $q * j(a) \leq j(q * a)$ and therefore $j(q * j(a)) \leq (j \circ j)(q * a) = j(q * a)$. On the other hand, $a \leq j(a)$ implies $q * a \leq q * j(a)$ and therefore $j(q * a) \leq j(q * j(a))$. $\square$

Corollary 5.3. Let j be a nucleus on a Q -algebra A and let $A_j = \{a \in A \mid j(a) = a\}$. Then $A_j = j^\rightarrow(A)$ and, moreover, A_j is a Q -algebra with the following structure:

- (i) $\bigvee_j S = j(\bigvee S)$ for every $S \subseteq A_j$;
- (ii) $a \otimes_j b = j(a \otimes b)$ for every $a, b \in A_j$;
- (iii) $q *_j a = j(q * a)$ for every $a \in A_j$ and $q \in Q$.

The map $A \xrightarrow{\nu} A_j$ given by: $\nu(a) = j(a)$ is an extremal epimorphism in $Q\text{-Alg}$.

Proof. Straightforward computations and Proposition 5.2 show that A_j with the aforesaid structure is a Q -algebra. As an example we show condition (ii) from Definition 2.7. Given $a, b \in A$ and $q \in Q$, $q *_j (a \otimes_j b) = j(q *_j (a \otimes b)) = j(q * (a \otimes b)) = j((q * a) \otimes b) = j(j(q * a) \otimes b) = (q *_j a) \otimes_j b$ and similarly $(q *_j a) \otimes_j b = a \otimes_j (q *_j b)$. The last statement follows from Proposition 5.2 and Corollary 3.5. $\square$

In [21] we have shown that every Q -algebra arises via a nucleus on a free Q -algebra.

6. FACTORIZATION OF Q -Alg-MORPHISMS

Bearing the concepts of the previous section in mind, in the current one we consider a method of producing subalgebras by means of extending a relation to a congruence. Start with some preliminary remarks.

Given a Q -algebra A , every $a \in A$ and $q \in Q$ give rise to the following adjunctions (in the sense of partially ordered sets as in [7]):

- $A \xrightleftharpoons[\cdot \otimes a]{a \rightarrow_l \cdot} A$ with $a \rightarrow_l b = \bigvee \{x \in A \mid x \otimes a \leq b\}$;
- $A \xrightleftharpoons[a \otimes \cdot]{a \rightarrow_r \cdot} A$ with $a \rightarrow_r b = \bigvee \{x \in A \mid a \otimes x \leq b\}$;
- $A \xrightleftharpoons[q * \cdot]{a \rightarrow * \cdot} Q$ with $q \rightarrow b = \bigvee \{x \in A \mid q * x \leq b\}$.

In case of a commutative quantale A the maps $a \rightarrow_l \cdot$ and $a \rightarrow_r \cdot$ coincide and will be denoted by $a \rightarrow \cdot$. The next proposition provides a characterization of the range of a nucleus.

Proposition 6.1. *Let A be a Q -algebra and let $S \subseteq A$. Equivalent are:*

- (i) *there exists a nucleus j on A such that $A_j = S$;*
- (ii) *if $T \subseteq S$, $s \in S$, $a \in A$ and $q \in Q$, then $\bigwedge T$, $a \rightarrow_l s$, $a \rightarrow_r s$, $q \rightarrow s \in S$.*

Proof. Implication (i) $\Rightarrow$ (ii) consists of straightforward computations. As an example we show that $j(q \rightarrow s) = q \rightarrow s$. Since $q * (q \rightarrow s) \leq s$, $q * j(q \rightarrow s) \leq j(q * (q \rightarrow s)) \leq j(s) = s$ and then $j(q \rightarrow s) \leq q \rightarrow s$. The other inequality is clear.

To prove implication (ii) $\Rightarrow$ (i) define $A \xrightarrow{j} A$ by: $j(a) = \bigwedge (\uparrow a \cap S)$. Then $j^{\rightarrow}(A) = S$. It is easy to see that j is a nucleus on A . As an example we show conditions (v) of Definition 5.1. Given $a \in A$ and $q \in Q$, $q * a \leq j(q * a)$ implies $a \leq q \rightarrow j(q * a)$ and therefore $j(a) \leq q \rightarrow j(q * a)$. It follows that $q * j(a) \leq j(q * a)$. $\square$

The following concept is motivated by the notion of saturated element in a frame (cf. Section 2.15 in [17]).

Definition 6.2. Let $R \subseteq A \times A$ be a binary relation on a Q -algebra A . An element $s \in A$ is called *saturated* (more precisely, *R -saturated*) provided that for every $a, b, c, d \in A$ and $q \in Q$, aRb implies:

- (i) $c \otimes a \leq s$ iff $c \otimes b \leq s$;
- (ii) $a \otimes c \leq s$ iff $b \otimes c \leq s$;
- (iii) $c \otimes a \otimes d \leq s$ iff $c \otimes b \otimes d \leq s$;
- (iv) $q * a \leq s$ iff $q * b \leq s$.

By A/R we denote the set of all saturated elements of A .

In the next proposition we show some properties of the aforesaid set A/R .

Proposition 6.3. *Let R be a binary relation on a Q -algebra A . If $S \subseteq A/R$, $t \in A/R$, $x \in A$ and $q \in Q$, then $\bigwedge S$, $x \rightarrow_l t$, $x \rightarrow_r t$, $q \rightarrow t \in A/R$.*

Proof. To give a better insight into the conditions of Definition 6.2 we provide the complete proof. Suppose $a, b, c \in A$ with aRb and $p \in Q$.

Since $c \otimes a \leq \bigwedge S$ iff $c \otimes a \leq s$ for $s \in S$ iff $c \otimes b \leq s$ for $s \in S$ iff $c \otimes b \leq \bigwedge S$ (conditions (ii) – (iv) of Definition 6.2 follow similarly), one gets $\bigwedge S \in A/R$.

Since $c \otimes a \leq x \rightarrow_l t$ iff $(c \otimes a) \otimes x \leq t$ iff $(c \otimes b) \otimes x \leq t$ iff $c \otimes b \leq x \rightarrow_l t$ (conditions (ii) – (iii) of Definition 6.2 follow similarly) and $p * a \leq x \rightarrow_l t$ iff $(p * a) \otimes x \leq t$ iff $a \otimes (p * x) \leq t$ iff $b \otimes (p * x) \leq t$ iff $p * b \leq x \rightarrow_l t$, one gets $x \rightarrow_l t \in A/R$. By analogy one shows that $x \rightarrow_r t \in A/R$.

Since $c \otimes a \leq q \rightarrow t$ iff $q * (c \otimes a) \leq t$ iff $(q * c) \otimes a \leq t$ iff $(q * c) \otimes b \leq t$ iff $c \otimes b \leq q \rightarrow t$ (conditions (ii) – (iii) of Definition 6.2 follow similarly) and $p * a \leq q \rightarrow t$ iff $q * (p * a) \leq t$ iff $(q \otimes p) * a \leq t$ iff $(q \otimes p) * b \leq t$ iff $p * b \leq q \rightarrow t$, one gets $q \rightarrow t \in A/R$. $\square$

Fix a Q -algebra A and a binary relation R on A . By Propositions 6.1 and 6.3 there exists a nucleus j on A with $j^\rightarrow(A) = A/R$ defined by: $j(a) = \bigwedge(\uparrow a \cap A/R)$. By Corollary 5.3 $A \xrightarrow{\nu_R} A/R$ given by: $\nu_R(a) = j(a)$ is an extremal epimorphism in $Q\text{-Alg}$. Moreover, the following lemma shows that ν_R respects R .

Lemma 6.4. *If $a, b \in A$ and aRb , then $\nu_R(a) = \nu_R(b)$.*

Proof. Since $a \leq \nu_R(a)$ and aRb , $b \leq \nu_R(a)$ and then $\nu_R(b) \leq (\nu_R \circ \nu_R)(a) = \nu_R(a)$. Similarly one gets $\nu_R(a) \leq \nu_R(b)$. $\square$

It is a matter of technique now to show that ν_R is a universal one among all $Q\text{-Alg}$ -morphisms with the property of Lemma 6.4 (cf. Section 2.15 in [17]).

Theorem 6.5. *Let R be a binary relation on a Q -algebra A . For every $Q\text{-Alg}$ -morphism $A \xrightarrow{h} B$ such that $h(a) = h(b)$ whenever aRb , there exists a unique $Q\text{-Alg}$ -morphism $A/R \xrightarrow{\bar{h}} B$ such that $\bar{h} \circ \nu_R = h$.*

Proof. Let $A \xrightarrow{h} B$ be a $Q\text{-Alg}$ -morphism such that $h(a) = h(b)$ whenever aRb and let $a \in A$. Define $\tau(a) = \bigvee\{x \in A \mid h(x) \leq h(a)\}$. Clearly $a \leq \tau(a)$ and $(h \circ \tau)(a) = h(a)$. Straightforward computations show that $\tau(a) \in A/R$. As an example check condition (i) from Definition 6.2. Take $a_1, a_2 \in A$ with a_1Ra_2 . Given $c \in A$, $c \otimes a_1 \leq \tau(a)$ implies $h(c \otimes a_2) = h(c) \otimes h(a_2) = h(c) \otimes h(a_1) = h(c \otimes a_1) \leq (h \circ \tau)(a) = h(a)$ and therefore $c \otimes a_2 \leq \tau(a)$. The converse implication follows similarly.

Since $\tau(a) \in A/R$, $a \leq \nu_R(a) \leq \tau(a)$. Thus $h(a) \leq (h \circ \nu_R)(a) \leq (h \circ \tau)(a) = h(a)$ and therefore $(h \circ \nu_R)(a) = h(a)$. Define $A/R \xrightarrow{\bar{h}} B$ by: $\bar{h}(a) = h(a)$. Then the triangle

$$\begin{array}{ccc} A & \xrightarrow{\nu_R} & A/R \\ & \searrow h & \downarrow \bar{h} \\ & & B \end{array}$$

commutes. By Lemma 3.6 $\bar{h}$ is a $Q\text{-Alg}$ -morphism. Uniqueness follows from the fact that ν_R is an epimorphism in $Q\text{-Alg}$. $\square$

Notice that by imposing some additional restrictions on the binary relation R (e.g., the condition of stability as in [17]) one can consider factorizations of $\mathbf{CSLat}(\vee)$ - as well as $\mathbf{Quant}$ -morphisms.

7. COPRODUCTS OF Q -ALGEBRAS

With the help of the already obtained results this section provides an explicit description of coproducts of Q -algebras. Let us start by recalling from [9] the construction of coproducts in the category $\mathbf{SGrp}$.

Remark 7.1. Suppose $(S_i)_{i \in I}$ is a family of semigroups. Let S be the set of all non-empty words $(s_1, s_2, \dots, s_n)$ with elements in $\bigcup_{i \in I} S_i$ such that two neighbouring elements are never in the same S_i . The semigroup operation “ $\cdot$ ” on S is defined as follows: if the last element x_n in x and the first element y_1 in y are in the same S_i , put $x \cdot y = x' * (x_n \cdot y_1) * y'$, where $*$ indicates concatenation and $x = x' * (x_n)$ and $y = (y_1) * y'$; otherwise, $x \cdot y = x * y$. For every $i \in I$ there exists a map $S_i \xrightarrow{\mu_i} S$ given by: $\mu_i(s) = (s)$. It is easy to see that the pair $((\mu_i)_{i \in I}, S)$ provides a coproduct of $(S_i)_{i \in I}$ in $\mathbf{SGrp}$.

The aforesaid remark together with Lemma 4.3, Remark 4.5 and Theorems 3.1, 6.5 imply the following result.

Remark 7.2. Suppose $(A_i)_{i \in I}$ is a family of Q -algebras. Let $((\mu_i)_{i \in I}, S)$ be a coproduct of $(VA_i)_{i \in I}$ in $\mathbf{SGrp}$. The sink $((\iota_i)_{i \in I}, Q^S/\mathbf{R})$ with $A_i \xrightarrow{\iota_i} Q^S/\mathbf{R} = A_i \xrightarrow{V\nu_{\mathbf{R}} \circ \eta_S \circ \mu_i} Q^S/\mathbf{R}$ is a coproduct of $(A_i)_{i \in I}$ in $Q\text{-Alg}$.

What are the elements of the Q -algebra $Q^S/\mathbf{R}$? The next proposition answers the question.

Proposition 7.3. *An element $\alpha \in Q^S$ is $\mathbf{R}$ -saturated iff for every $i \in I$ and $x, y \in S$ the following hold:*

- (1) *given $T \subseteq A_i$:*
 - (i) $\alpha(x \cdot (\bigvee T)) = \bigwedge_{t \in T} \alpha(x \cdot (t));$
 - (ii) $\alpha((\bigvee T) \cdot x) = \bigwedge_{t \in T} \alpha((t) \cdot x);$
 - (iii) $\alpha(x \cdot (\bigvee T) \cdot y) = \bigwedge_{t \in T} \alpha(x \cdot (t) \cdot y);$
 - (iv) $\alpha((\bigvee T)) = \bigwedge_{t \in T} \alpha((t)).$
- (2) *given $a \in A_i$ and $q \in Q$:*
 - (i) $\alpha(x \cdot (q * a)) = q \rightarrow \alpha(x \cdot (a));$
 - (ii) $\alpha((q * a) \cdot x) = q \rightarrow \alpha((a) \cdot x);$
 - (iii) $\alpha(x \cdot (q * a) \cdot y) = q \rightarrow \alpha(x \cdot (a) \cdot y);$
 - (iv) $\alpha((q * a)) = q \rightarrow \alpha((a)).$

Proof. The proof consists of straightforward computations. As an example we show condition (1).(i). Suppose $i \in I$, $T \subseteq A_i$ and $\beta \in Q^S$. By condition (i) of Definition 6.2 we have to show that $\beta \otimes ((\eta_S \circ \mu_i)(\bigvee T)) \leq \alpha$ iff $\beta \otimes (\bigvee (\eta_S \circ \mu_i)^{\rightarrow}(T)) \leq \alpha$. Notice that $\alpha \geq \beta \otimes ((\eta_S \circ \mu_i)(\bigvee T))$ iff $\alpha(x) \geq (\beta \otimes ((\eta_S \circ \mu_i)(\bigvee T)))(x) = \bigvee_{x=y \cdot z} \beta(y) \otimes ((\eta_S \circ \mu_i)(\bigvee T))(z) = \bigvee_{x=y \cdot (\bigvee T)} \beta(y)$ for $x \in S$. Similarly $\alpha \geq \beta \otimes$

$(\bigvee(\eta_S \circ \mu_i)^{\rightarrow}(T))$ iff $\alpha(x) \geq \beta \otimes (\bigvee(\eta_S \circ \mu_i)^{\rightarrow}(T))(x) = \bigvee_{t \in T} \bigvee_{x=y \cdot (t)} \beta(y)$ for $x \in S$.

It is a matter of technique now to show that the following are equivalent:

- (i) given $\beta \in Q^S$, $\bigvee_{x=y \cdot (\bigvee T)} \beta(y) \leq \alpha(x)$ for $x \in S$ iff $\bigvee_{t \in T} \bigvee_{x=y \cdot (t)} \beta(y) \leq \alpha(x)$ for $x \in S$;
- (ii) $\alpha(x \cdot (\bigvee T)) = \bigwedge_{t \in T} \alpha(x \cdot (t))$ for $x \in S$. □

In case of the category **2-Alg** we get a description of coproducts in the category **Quant** (drop item (2) in Proposition 7.3) as the collection of all sets $\alpha \in \mathcal{P}(S)$ such that given $x, y \in S$ (here possibly also void) and $T \subseteq A_i$ for some $i \in I$, $x \cdot (\bigvee T) \cdot y \in \alpha$ iff $x \cdot (t) \cdot y \in \alpha$ for every $t \in T$ (in particular, $x \in \alpha$ for every $x = (b_1, \dots, b_{i-1}, \perp, b_{i+1}, \dots, b_n) \in S$). A similar characterization has been already obtained in [5]. Notice, however, that [5] does not provide an explicit description of the elements of the coproduct in question.

8. CONCLUDING EXAMPLE

To give a better insight into the structure of the set $Q^S/\mathbf{R}$ we will show an example of an **R**-saturated element. Start with some preliminary results.

Given a Q -algebra A , every $a \in A$ gives rise to the adjunction $A \xrightleftharpoons[\cdot * a]{a \leftarrow \cdot} Q$ with $a \leftarrow b = \bigvee \{q \in Q \mid q * a \leq b\}$ (notice that for $Q = \mathbf{2}$ the map $\cdot \leftarrow a$ is the lower set $\downarrow a = \{b \in A \mid b \leq a\}$). The adjunction provides a Q -pseudocomplementation on A , i.e., a map $A \xrightarrow{(-)^*} Q$ defined by: $a^* = a \leftarrow \perp$. It is easy to see that $q \leq a^*$ iff $q * a = \perp$ for every $q \in Q$ and $a \in A$.

Proposition 8.1. *Let A be a Q -algebra. If $a, b, c \in A$, $T \subseteq A$ and $q \in Q$, then:*

- (i) $(\bigvee T) \leftarrow a = \bigwedge_{t \in T} (t \leftarrow a)$;
- (ii) $(q * b) \leftarrow a = q \rightarrow (b \leftarrow a)$;
- (iii) $b \leftarrow (c \rightarrow_l a) = (b \otimes c) \leftarrow a = c \leftarrow (b \rightarrow_r a)$;
- (iv) $\top \otimes a^* = a^*$.

Proof. Straightforward computations show the aforesaid properties. As an example we show (i). Given $t \in T$, $((\bigvee T) \leftarrow a) * t \leq ((\bigvee T) \leftarrow a) * (\bigvee T) \leq a$ and then $(\bigvee T) \leftarrow a \leq t \leftarrow a$. Thus $(\bigvee T) \leftarrow a \leq \bigwedge_{t \in T} (t \leftarrow a)$. On the other hand, $(\bigwedge_{t \in T} (t \leftarrow a)) * (\bigvee T) = \bigvee_{t' \in T} (\bigwedge_{t \in T} (t \leftarrow a)) * t' \leq \bigvee_{t' \in T} (t' \leftarrow a) * t' \leq a$ and then $\bigwedge_{t \in T} (t \leftarrow a) \leq (\bigvee T) \leftarrow a$. □

Recall the notations of Remark 7.2. Given $i \in I$, every $a \in A_i$ provides a map $S \xrightarrow{\alpha^a} Q$ defined by: $\alpha^a(x) = b \leftarrow a$, if $x = (b)$ with $b \in A_i$; otherwise, $\alpha^a(x) = \top$. We also set $\alpha_a = (\eta_S \circ \mu_i)(a)$.

Lemma 8.2.

- (i) $\alpha^a \in Q^S/\mathbf{R}$;
- (ii) $\alpha_a \leq \alpha^a$;
- (iii) $\nu_{\mathbf{R}}(\alpha_a) \leq \alpha^a$;

- (iv) given $\beta \in Q^S/\mathbf{R}$, $\bigvee_{i=1}^n b_i^* \leq \beta(x)$ for every $x = (b_1, \dots, b_n) \in S$; if $\alpha_a \leq \beta$, then $\alpha^a((b)) \leq \beta((b))$ for every $b \in A_i$.

Proof.

- (i) The proof consists of straightforward computations. As an example we show condition (1).(i) of Proposition 7.3. If $x = (b)$ with $b \in A_i$ and $T \subseteq A_i$, then $\alpha^a(x \cdot (\bigvee T)) = b \otimes (\bigvee T) \leftarrow a = (\bigvee T) \leftarrow (b \rightarrow_r a) = \bigwedge_{t \in T} (t \leftarrow (b \rightarrow_r a)) = \bigwedge_{t \in T} ((b \otimes t) \leftarrow a) = \bigwedge_{t \in T} \alpha^a(x \cdot (t))$ by Proposition 8.1; otherwise, $\alpha^a(x \cdot (\bigvee T)) = \top = \bigwedge_{t \in T} \alpha^a(x \cdot (t))$.
- (ii) Clear.
- (iii) Follows from (i) and (ii).
- (iv) For $i \in I$, $q \in Q$ with $q * b_i = \perp$, $\top = \beta((b_1, \dots, b_{i-1}, q * b_i, b_{i+1}, \dots, b_n)) = q \rightarrow \beta(x)$ by Proposition 7.3 and therefore $\top \otimes q \leq \beta(x)$. Thus $b_i^* \leq \beta(x)$ for every $i \in I$.
Let $b \in A_i$. Given $q \in Q$ with $q * b \leq a$, $1 = \alpha_a((a)) \leq \beta((a)) \leq \beta((q * b)) = q \rightarrow \beta((b))$ by Proposition 7.3 and then $q \leq \beta((b))$. The result follows. $\square$

In view of Lemma 8.2 one can suggest that $\nu_{\mathbf{R}}(\alpha_a) = \alpha^a$. Unfortunately, that is not the case since for the category $\mathbf{2-Alg}$, $\alpha^a = S \setminus \{(b) \mid b \in A_i : b \not\leq a\}$, whereas $\nu_{\mathbf{R}}(\alpha_a) = \{(b) \mid b \in A_i : b \leq a\} \cup \{x \mid x \in S : x = (b_1, \dots, b_{i-1}, \perp, b_{i+1}, \dots, b_n)\}$. One can try to define a map $S \xrightarrow{\beta^a} Q$ by: $\beta^a(x) = \alpha^a(x)$, if $x = (b)$ with $b \in A_i$; otherwise, $\beta^a(x) = \bigvee_{i=1}^n b_i^*$ with $x = (b_1, \dots, b_n)$. It is still an open problem whether the map β^a is $\mathbf{R}$ -saturated.

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