

CONVEXITY AND COCONVEXITY OF S -POSETS

LAURI TART

ABSTRACT. We will give a characterization of convex subobjects in the category of S -posets, based on weighted limits. We also investigate how this treatment gives rise to the dual concept of coconvex factor objects, and prove that these amount to S -poset isomorphisms. Finally, we will apply this description to prove that there are no dual equivalences between S -poset categories.

1. INTRODUCTION

The notion of convexity, or rather of convex subobjects, is of some importance in the study of S -posets. In particular, it turns out to be important to consider such subobjects in the context of Morita duality for S -posets. One might think that a Morita duality would turn convex subobjects into their duals, which we will call coconvex factor objects. This is unfortunately not the case, since coconvex factor objects turn out to be rather limited.

We will denote the category of all partial orders and monotone maps by $\mathbf{Pos}$. This category is a symmetric monoidal closed category. For an introduction to enriched categories in general, we refer the reader to [4]. At this point we will only mention that a $\mathbf{Pos}$ -category (i.e. a category enriched over $\mathbf{Pos}$) is a category where the morphisms between any pair of objects form a poset and their composition preserves this order in both arguments. We recall that the category of all partially ordered right (left) acts over a fixed partially ordered monoid (pomonoid) S forms a $\mathbf{Pos}$ -category (with morphisms being monotone right (left) act homomorphisms, ordered pointwise). For a given pomonoid S , the corresponding $\mathbf{Pos}$ -category will be denoted $\mathbf{Pos}_S$ (${}_S\mathbf{Pos}$) and its objects will be called *right (left) S -posets*.

A functor $F : \mathbf{Pos}_S \rightarrow \mathbf{Pos}_T$ is a *Pos-functor* if it also preserves the order of morphisms, that is, the assignments $F : \mathbf{Pos}_S(A, B) \rightarrow \mathbf{Pos}_T(F(A), F(B))$ are monotone for all $A, B \in \mathbf{Pos}_S$. This means that for every pair of S -poset morphisms $f_1, f_2 : A \rightarrow B$ with $f_1(x) \leq f_2(x)$ for all $x \in A$, also $F(f_1)(y) \leq F(f_2)(y)$ for all $y \in F(A)$.

A *Pos-natural transformation* $\mu : F \rightarrow G : \mathbf{Pos}_S \rightarrow \mathbf{Pos}_T$ is a family of right T -poset morphisms $(\mu_A : F(A) \rightarrow G(A))_{A \in \mathbf{Pos}_S}$, subject to the usual naturality condition.

Two $\mathbf{Pos}$ -functors $F : \mathbf{Pos}_S \rightarrow \mathbf{Pos}_T$ and $G : \mathbf{Pos}_T \rightarrow \mathbf{Pos}_S$ are called *Pos-equivalence functors* if $1_{\mathbf{Pos}_T} \cong F \circ G$ and $1_{\mathbf{Pos}_S} \cong G \circ F$, and these natural isomorphisms are $\mathbf{Pos}$ -natural, i.e. the components are order isomorphisms. Correspondingly, the categories $\mathbf{Pos}_S$ and $\mathbf{Pos}_T$ are called *Pos-equivalent* if there exist $\mathbf{Pos}$ -equivalence functors $F : \mathbf{Pos}_S \rightarrow \mathbf{Pos}_T$ and $G : \mathbf{Pos}_T \rightarrow \mathbf{Pos}_S$. There is also the

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notion of *dual Pos-equivalence*, which means there are functors $F : \mathbf{Pos}_S \rightarrow {}_T\mathbf{Pos}$ and $G : {}_T\mathbf{Pos} \rightarrow \mathbf{Pos}_S$, which satisfy conditions $1_{{}_T\mathbf{Pos}} \cong F \circ G$ and $1_{\mathbf{Pos}_S} \cong G \circ F$, and those natural isomorphisms are **Pos**-natural.

For an S -poset A and a binary relation $\rho \subseteq A \times A$, define the relation

$$a \leq_\rho a'$$

to hold if there exist $a_1, a'_1, \dots, a_n, a'_n \in A$ such that

$$a \leq a_1 \rho a'_1 \leq \dots \leq a_n \rho a'_n \leq a'.$$

For an S -poset A , an equivalence relation $\rho \subseteq A \times A$ is called an *S -poset congruence*, if it is an S -act congruence (meaning $as \rho a's$ whenever $a \rho a', a, a' \in A, s \in S$) and it satisfies the *closed chains condition*: if

$$a \leq_\rho a' \leq_\rho a,$$

then also

$$a \rho a'.$$

Factoring by an S -poset congruence gives us a *factor S -poset*, if we define the order on equivalence classes by

$$[a]_\rho \leq [a']_\rho \iff a \leq_\rho a'.$$

For an S -poset B , a sub- S -poset A of an S -poset B is called *convex* if for all $b \in B$, $a_1, a_2 \in A$, whenever $a_1 \leq b \leq a_2$, then also $b \in A$. We call an S -poset morphism $f : A \rightarrow B$ *convex* if $f(A)$ is a convex sub- S -poset of B .

We recall from Section 2.3 of [2] that in the category of S -posets monomorphisms are precisely injective S -poset morphisms, regular monomorphisms (equalizers) are exactly order-reflecting S -poset morphisms (embeddings). Similarly, Section 2.2 of [2] shows that in the category of S -posets, epimorphisms are surjective S -poset morphisms, regular epimorphisms are epimorphisms $\pi : A \rightarrow B$ for which whenever $\pi(a) \leq \pi(a')$ in B , then also $a \leq_{\ker \pi} a'$. The latter fact is equivalent to saying that regular epimorphisms are exactly these morphisms that correspond to factor objects in the sense defined above.

2. CONVEXITY

We will first see that convex regular subobjects admit a nice description via the so-called weighted limits (for the definition and a detailed discussion of weighted limits and colimits for S -posets, see [3]). Then we will dualize the notion to get what we will call coconvex factor objects.

While not exactly the definition of a weighted limit, Theorem 3 of [3] gives necessary and sufficient conditions for the existence of one and we will adopt these conditions as a definition, changing left S -posets into right S -posets.

Definition 2.1. For a pomonoid S , a small **Pos**-category $\mathcal{A}$ and **Pos**-functors $F : \mathcal{A} \rightarrow \mathbf{Pos}_S$ and $G : \mathcal{A} \rightarrow \mathbf{Pos}$, a right S -poset L together with a family $(l_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ of right S -poset morphisms $l_A^x : L \rightarrow F(A)$, called *projections*, is called a *limit of F weighted by G* , denoted by $\lim_G F$, if the following conditions hold

- (1) (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$

$$x \leq x' \implies l_A^x \leq l_A^{x'};$$

(b) for all $a : A_0 \rightarrow A_1$ in $\mathcal{A}$ and $x \in G(A_0)$,

$$F(a) \circ l_{A_0}^x = l_{A_1}^{G(a)(x)};$$

(2) for every $P \in \mathbf{Pos}_S$ and family $(p_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ of right S -poset morphisms $p_A^x : P \rightarrow F(A)$ with property (1), there is a unique right S -poset morphism $\varphi : P \rightarrow L$ such that $l_A^x \circ \varphi = p_A^x$ for every $A \in \mathcal{A}$ and $x \in G(A)$.

Fix a regular subobject (i.e. a regular monomorphism) $f : A \rightarrow B$ in $\mathbf{Pos}_S$. Take the $\mathbf{Pos}$ -category $\mathcal{A} = \mathbf{2}$, where $\mathbf{2}$ is the 2-element chain $\{0, 1\}$. Then we have $\mathbf{Pos}$ -functors $F : \mathcal{A} \rightarrow \mathbf{Pos}_S$ and $G : \mathcal{A} \rightarrow \mathbf{Pos}$, with $F(0 < 1) = f$, $G(0 < 1) = c_{02} : \mathbf{2} \rightarrow \mathbf{3}$, where $\mathbf{3}$ is the 3-element chain $\{0, 1, 2\}$ and $c_{02}(0) = 0$, $c_{02}(1) = 2$.

$$\begin{array}{ccc} F(\mathcal{A}) : A & \xrightarrow{f} & B \\ \mathcal{A} : 0 & \longrightarrow & 1 \\ G(\mathcal{A}) : \mathbf{2} & \xrightarrow{c_{02}} & \mathbf{3} \end{array}$$

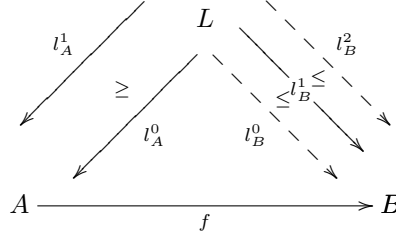
Definition 2.2. For the above special functors F and G , we call the limit of F weighted by G the *sandwich limit* of f .

Remark 2.3. Applying the conditions of the preceding definition, with $\mathcal{A}, F, G$ defined as above, we see that the sandwich limit of f is a right S -poset L together with S -poset morphisms $l_A^0, l_A^1 : L \rightarrow A$, $l_B^1, l_B^2 : L \rightarrow B$ such that

1. $l_A^0 \leq l_A^1$, $f \circ l_A^0 \leq l_B^1 \leq f \circ l_A^1$;
2. for every $P \in \mathbf{Pos}_S$ and $p_A^0, p_A^1 : P \rightarrow A$, $p_B^1 : P \rightarrow B$ that satisfy property 1, there is exactly one right S -poset morphism $\varphi : P \rightarrow L$ such that

$$l_A^i \circ \varphi = p_A^i, \quad l_B^1 \circ \varphi = p_B^1$$

for $i = 0, 1$.



Remark 2.4. By Section 2.3 of [3], the limit of F weighted by G admits a canonical construction as the (componentwise ordered) right S -poset

$$L = \{f = (f_A)_{A \in \mathcal{A}} \in \prod_{A \in \mathcal{A}} \mathbf{Pos}(G(A), F(A)) \mid F(a) \circ f_{A_0} = f_{A_1} \circ G(a) \text{ for all } a : A_0 \rightarrow A_1 \text{ in } \mathcal{A}\}.$$

The corresponding projections $l_A^x : L \rightarrow F(A)$ are defined as

$$l_A^x((f_A)_{A \in \mathcal{A}}) = f_A(x),$$

for $A \in \mathcal{A}$, $x \in G(A)$.

Using this construction, the sandwich limit of f can be constructed as

$$\begin{aligned}
L &= \{(f_A)_{A \in \mathcal{A}} | f_A : G(A) \rightarrow F(A), F(a) \circ f_{A_0} = f_{A_1} \circ G(a) \\
&\quad \text{for all } a : A_0 \rightarrow A_1 \text{ in } \mathcal{A}\} \\
&= \{(f_0, f_1) | f_0 : \mathbf{2} \rightarrow A, f_1 : \mathbf{3} \rightarrow B, f \circ f_0 = f_1 \circ c_{02}\} \\
&\cong \{((x_0, x_1), (y_0, y_1, y_2)) \in A^2 \times B^3 | x_0 \leq x_1, y_0 \leq y_1 \leq y_2, \\
&\quad f(x_0) = y_0, f(x_1) = y_2\} \\
&\cong \{(x_0, y, x_1) \in A \times B \times A | f(x_0) \leq y \leq f(x_1)\}.
\end{aligned}$$

Note that the last isomorphism uses the fact that f as a regular monomorphism reflects order.

The projections turn out to be simply

$$\begin{aligned}
l_A^0(x_0, y, x_1) &= f_0(0) = x_0; \\
l_A^1(x_0, y, x_1) &= f_0(1) = x_1; \\
l_B^1(x_0, y, x_1) &= f_1(1) = y.
\end{aligned}$$

We can therefore conclude the preceding with the following result.

Theorem 2.5. *A regular subobject $f : A \rightarrow B$ is convex if and only if the pullback of f along the projection l_B^1 of the sandwich limit of f is an epimorphism.*

$$\begin{array}{ccc}
A \times_B \lim_G F & \xrightarrow{\pi} & \lim_G F \\
\downarrow & & \downarrow l_B^1 \\
A & \xrightarrow{f} & B
\end{array}$$

Proof. We will use the construction of $\lim_G F = L$ as given in Remark 2.4. Canonically, the pullback of f and l_B^1 is $A \times_B \lim_G F = \{(x, x_0, y, x_1) \in A^2 \times B \times A | f(x) = l_B^1(x_0, y, x_1) = y, f(x_0) \leq y \leq f(x_1)\}$. Therefore the pullback of f along $l_B^1 : \lim_G F \rightarrow B$ is the projection $\pi : A \times_B \lim_G F \rightarrow \lim_G F$. Requiring this pullback π to be an epimorphism means that for all $(x_0, y, x_1) \in A \times B \times A$ such that $f(x_0) \leq y \leq f(x_1)$ there must exist an $x \in A$ such that $f(x) = y$, which is the definition of $f(A)$ being convex in B . $\square$

3. COCONVEXITY

For the dual situation, the essential step is to construct the corresponding weighted colimit.

First, we dualize Definition 2.1 to get a practical definition for weighted colimits.

Definition 3.1. For a pomonoid S , a small Pos-category $\mathcal{A}$ and Pos-functors $F : \mathcal{A} \rightarrow \text{Pos}_S$ and $G : \mathcal{A}^{\text{op}} \rightarrow \text{Pos}$, a right S -poset L together with a family $(l_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ of right S -poset morphisms $l_A^x : F(A) \rightarrow L$, called *injections*, is called a *colimit of F weighted by G* , denoted by $\text{colim}_G F$, if the following conditions hold

- (1) (a) for all $A \in \mathcal{A}$ and $x, x' \in G(A)$

$$x \leq x' \implies l_A^x \leq l_A^{x'};$$

(b) for all $a : A_0 \rightarrow A_1$ in $\mathcal{A}$ and $x \in G(A_1)$,

$$l_{A_1}^x \circ F(a) = l_{A_0}^{G(a)(x)};$$

(2) for every $P \in \mathbf{Pos}_S$ and family $(p_A^x)_{A \in \mathcal{A}}^{x \in G(A)}$ of right S -poset morphisms $p_A^x : F(A) \rightarrow P$ with property (1), there is a unique right S -poset morphism $\varphi : L \rightarrow P$ such that $\varphi \circ l_A^x = p_A^x$ for every $A \in \mathcal{A}$ and $x \in G(A)$.

Fix a factor object (i.e. a regular epimorphism) $f : A \rightarrow B$ in $\mathbf{Pos}_S$. Take the index category $\mathcal{A} = \{0 > 1\}$, a functor $F : \mathcal{A} \rightarrow \mathbf{Pos}_S$ as $F(1 < 0) = f$, and let $G : \mathcal{A}^{\text{op}} \rightarrow \mathbf{Pos}$ be the same functor as it was for convexity.

$$\begin{array}{ccc} F(\mathcal{A}) : A & \xrightarrow{f} & B \\ \mathcal{A} : 1 & \longrightarrow & 0 \\ G(\mathcal{A}) : \mathbf{3} & \xleftarrow{c_{02}} & \mathbf{2} \end{array}$$

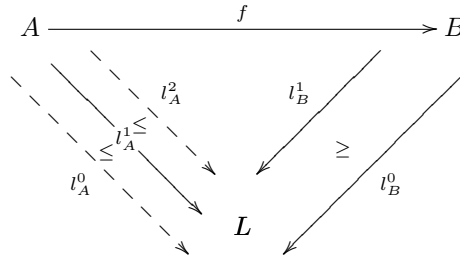
Definition 3.2. For the above special functors F and G , the colimit of F weighted by G is called the *sandwich colimit* of f .

Remark 3.3. Again, using the conditions of the first definition, with $\mathcal{A}$, F and G as defined above, we see that the sandwich colimit of f is a right S -poset L together with three S -poset morphisms $l_B^0, l_B^1 : B \rightarrow L$, $l_A^1 : A \rightarrow L$ such that

1. $l_B^0 \leq l_B^1$, $l_B^0 \circ f \leq l_A^1 \leq l_B^1 \circ f$;
2. for every $P \in \mathbf{Pos}_S$ and $p_B^0, p_B^1 : B \rightarrow P$, $p_A^1 : A \rightarrow P$ that satisfy property 1, there is precisely one right S -poset morphism $\varphi : P \rightarrow L$ such that

$$\varphi \circ l_B^i = p_B^i, \quad \varphi \circ l_A^1 = p_A^1$$

for $i = 0, 1$.



Remark 3.4. The sandwich colimit of f can be constructed as follows (for the general construction of S -poset colimits, see [3]).

For L , take $(\mathbf{2} \times B) \sqcup A$, retain the action on components as it is done with S -act colimits and define the following binary relation $\prec$ on L :

- (1) $(i, b) \prec (j, b')$ for all $b \leq b'$ in B and $i \leq j$, $i, j \in \{0, 1\}$;
- (2) $a \prec a'$ for all $a \leq a'$ in A ;
- (3) $(0, b) \prec a$ whenever $b \leq f(a)$ in B ;
- (4) $a \prec (1, b)$ whenever $f(a) \leq b$ in B .

Since A and B are S -posets, $\prec$ is clearly reflexive. As there are only order relations in one direction between different disjoint components, and the inherited orders on the components are antisymmetric, $\prec$ is also antisymmetric. Finally, if $x \prec y \prec z$, then we have the following possibilities:

- a) $x = (i, b)$, $y = (j, b')$, $z = (k, b'')$ and $b \leq b' \leq b''$, $i \leq j \leq k$, so $b \leq b''$, $i \leq k$ and $x \prec z$ by (1);
- b) $x = (i, b)$, $y = (j, b')$, $z = a''$ and $b \leq b'$, $b' \leq f(a'')$, $i \leq j = 0$, so $b \leq f(a'')$, $i = 0$ and $x \prec z$ by (3);
- c) $x = (i, b)$, $y = a'$, $z = (k, b'')$ and $b \leq f(a') \leq b''$, $i = 0$, $k = 1$, so $b \leq b''$, $i \leq k$ and $x \prec z$ by (1);
- d) $x = (i, b)$, $y = a'$, $z = a''$ and $b \leq f(a')$, $a' \leq a''$, $i = 0$, so $b \leq f(a') \leq f(a'')$, $i = 0$ and $x \prec z$ by (3);
- e) $x = a$, $y = (j, b')$, $z = (k, b'')$ and $f(a) \leq b' \leq b''$, $1 = j \leq k$, so $f(a) \leq b''$, $k = 1$ and $x \prec z$ by (4);
- f) $x = a$, $y = (j, b')$, $z = a''$ and $f(a) \leq b' \leq f(a'')$, $j = 0$, $j = 1$ and this situation never occurs;
- g) $x = a$, $y = a'$, $z = (k, b'')$ and $a \leq a'$, $f(a') \leq b''$, $k = 1$, so $f(a) \leq f(a') \leq b''$, $k = 1$ and $x \prec z$ by (4);
- h) $x = a$, $y = a'$, $z = a''$ and $a \leq a' \leq a''$, so $a \leq a''$ and $x \prec z$ by (2).

Thus in any case we have $x \prec z$ and we have a partial order on L .

This L is clearly an S -act, and we must check whether its action preserves order.

First, if we have $s \leq t$ in S , then $as \leq at$, $(i, b)s = (i, bs) \leq (i, bt) = (i, b)t$ for $a \in A$, $b \in B$, $i = 1, 2$.

Second, for cases (1) and (2), let $s \in S$ and either $a \leq a'$ or $(i, b) \leq (j, b')$ for $a \in A$, $b \in B$, $i \in \{0, 1\}$; then $as \leq a's$ or $(i, b)s = (i, bs) \leq (j, b's) = (j, b)s$ since $i \leq j$ and $bs \leq b's$. In case (3) the relation is $(0, b) \prec a$, i.e. $b \leq f(a)$, so for $s \in S$ we have $bs \leq f(a)s = f(as)$ and $(0, bs) \prec as$. In case (4) there is $a \prec (1, b)$, i.e. $f(a) \leq b$, so for $s \in S$ we have $f(as) = f(a)s \leq bs$ and $as \prec (1, bs)$. Therefore L is an S -poset.

We furnish L with the usual coproduct injections $l_A^1 : A \rightarrow L$ and $l_B^0, l_B^1 : B \rightarrow L$.

Due to the definition of the order relation $\prec$, we have $(0, b) \prec (1, b)$ for all $b \in B$, so also $l_B^0 \leq l_B^1$. On the same note, $(0, f(a)) \prec a \prec (1, f(a))$ for every $a \in A$, so $l_B^0 \circ f \leq l_A^1 \leq l_B^1 \circ f$. This proves that condition 1 of Remark 3.3 holds.

Take $P \in \text{Pos}_S$ and $p_B^0, p_B^1 : B \rightarrow P$, $p_A^1 : A \rightarrow P$ which satisfy property 1 of Remark 3.3. Define a morphism $\varphi : L \rightarrow P$ by taking

$$\varphi((i, b)) = p_B^i(b), \quad i = 0, 1; \quad \varphi(a) = p_A^1(a).$$

As any p_X^i is always an S -poset morphism, we must only check the preservation of order relations of types (3), (4) and the case $(0, b) \prec (1, b')$, with $b \leq b'$. Take $(0, b) \prec a$, then $b \leq f(a)$ and

$$\varphi((0, b)) = p_B^0(b) \leq p_B^0(f(a)) \leq p_A^1(a) = \varphi(a).$$

The case for (4) is symmetric. For $(0, b) \prec (1, b')$ we have $b \leq b'$ and

$$\varphi((0, b)) = p_B^0(b) \leq p_B^1(b) \leq p_B^1(b') = \varphi((1, b')).$$

Thus φ is an S -poset morphism.

To see that this is the unique way to define φ , consider an S -poset morphism $\psi : L \rightarrow P$ which satisfies $\psi \circ l_A^1 = p_A^1$ and $\psi \circ l_B^i = p_B^i$, $i \in \{0, 1\}$. Then

$$\psi(a) = (\psi \circ l_A^1)(a) = p_A^1(a) = \varphi(a)$$

and

$$\psi((i, b)) = (\psi \circ l_B^i)(b) = p_B^i(b) = \varphi((i, b)), \quad i \in \{0, 1\}.$$

Therefore $\psi = \varphi$.

Finally, $\varphi \circ l_A^1 = p_A^1$, $\varphi \circ l_B^i = p_B^i$, $i = 0, 1$ by construction.
 This completes the proof that L is the sandwich colimit of f .

Now we can dualize Theorem 2.5 and define coconvexity.

Definition 3.5. We call a regular epimorphism $f : A \rightarrow B$ *coconvex* if the pushout of f along the middle injection $l_A^1 : A \rightarrow \text{colim}_G F$ of the sandwich colimit of f is a monomorphism.

Proposition 3.6. *Coconvex regular epimorphisms in Pos_S are isomorphisms.*

Proof. Note that due to naming conventions, we have $l_A^1(a) = a$ for all $a \in A$.

By Remark 7.70(3) of [1], regular epimorphisms which are also monomorphisms are isomorphisms, so we only need to show that f is a monomorphism.

We shall see that by defining $\iota : \text{colim}_G F \rightarrow \mathbf{3} \times B$ and $\iota' : B \rightarrow \mathbf{3} \times B$ with

$$\iota((i, b)) = (c_{02}(i), b), \quad i = 0, 1, \quad b \in B, \quad \iota(a) = (1, f(a)), \quad a \in A$$

and

$$\iota'(b) = (1, b), \quad b \in B,$$

we obtain a pushout square

$$\begin{array}{ccc} \mathbf{3} \times B & \xleftarrow{\iota} & \text{colim}_G F \\ \iota' \uparrow & & \uparrow l_A^1 \\ B & \xleftarrow{f} & A \end{array} \quad .$$

First, ι' as the middle injection of a coproduct is clearly an S -poset homomorphism.

Second, for $s \in S$ and $a \in A$, $b \in B$, $i \in \{0, 1\}$ we have

$$\iota(a)s = (1, f(a))s = (1, f(as)) = \iota(as),$$

$$\iota((i, b))s = (c_{02}(i), b)s = (c_{02}(i), bs) = \iota((i, bs)),$$

so ι is an S -act morphism. Further, if $x \prec y$ in $\text{colim}_G F$, then we have the following possibilities:

- 1) $x = (i, b)$, $y = (j, b')$, $b \leq b'$ in B and $i \leq j$, and we get

$$\iota(x) = (c_{02}(i), b) \leq (c_{02}(j), b') = \iota(y);$$

- 2) $x = a$, $y = a'$, $a \leq a'$ in A , and we get

$$\iota(x) = (1, f(a)) \leq (1, f(a')) = \iota(y);$$

- 3) $x = (0, b)$, $y = a$, $b \leq f(a)$ in B , and we get

$$\iota(x) = (0, b) \leq (1, f(a)) = \iota(y);$$

- 4) $x = a$, $y = (1, b)$, $f(a) \leq b$ in B , and we get

$$\iota(x) = (1, f(a)) \leq (2, b) = \iota(y).$$

So ι preserves order and is thus an S -poset morphism.

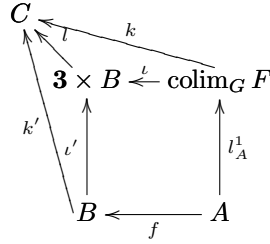
We can easily compute that for $a \in A$,

$$(\iota \circ l_A^1)(a) = \iota(l_A^1(a)) = \iota(a) = (1, f(a)) = \iota'(f(a)) = (\iota' \circ f)(a).$$

This makes our square commutative.

Finally, assume we have an S -poset C and two S -poset morphisms $k: \text{colim}_G F \rightarrow C$ and $k': B \rightarrow C$ such that

$$k' \circ f = k \circ l_A^1.$$



We shall define a map $l: \mathbf{3} \times B \rightarrow C$ with

$$l((0, b)) = k((0, b)), \quad l((1, b)) = k'(b), \quad l((2, b)) = k((1, b))$$

for $b \in B$. We can see that since

$$l((0, b)) = (l \circ \iota)((0, b)) = k((0, b)),$$

$$l((1, b)) = (l \circ \iota')(b) = k'(b),$$

$$l((2, b)) = (l \circ \iota)((1, b)) = k((1, b)),$$

this is the only definition that could provide us with $l \circ \iota = k$ and $l \circ \iota' = k'$. But we have

$$\begin{aligned} (l \circ \iota)(a) &= (l \circ (\iota \circ l_A^1))(a) = (l \circ (\iota' \circ f))(a) \\ &= l((1, f(a))) = k'(f(a)) = k(l_A^1(a)) = k(a), \end{aligned}$$

and such a definition does provide us with these equalities.

Since k and k' are compatible with S -action, so is l . For order-preservation, we can split $(i, b) \leq (j, b')$ (then $i \leq j$, $b \leq b'$) into the following cases:

- a) $i = j$, and $l((i, b)) \leq l((j, b'))$, since k and k' preserve order;
- b) $i = 0$, $j = 1$ and $l((0, b)) = k((0, b)) \leq k(a) = (k \circ l_A^1)(a) = (k' \circ f)(a) = k'(b') = l((1, b'))$, since $b \leq b' = f(a)$ for some $a \in A$, and so $(0, b) \prec a$;
- c) $i = 0$, $j = 2$ and $l((0, b)) = k((0, b)) \leq k((1, b')) = l((2, b'))$;
- d) $i = 1$, $j = 2$ and $l((1, b)) = k'(b) = (k' \circ f)(a) = (k \circ l_A^1)(a) = k(a) \leq k((1, b')) = l((2, b'))$, since $f(a) = b \leq b'$ for some $a \in A$, and so $a \prec (1, b')$.

Therefore l is an S -poset morphism, it satisfies $l \circ \iota = k$ and $l \circ \iota' = k'$ and is defined in the only way that these equations can be obtained. So $\mathbf{3} \times B$ is indeed the pushout of f and l_A^1 .

And finally, looking at the definition of ι , we can see that requiring ι to be a monomorphism (i.e. injective) is equivalent to requiring f to be a monomorphism (i.e. injective), since $\iota(a) = (1, f(a))$, the image is separated into three disjoint components and on other components ι is always injective. $\square$

4. APPLICATION

The fact that the duals of convex regular subobjects in the category of S -posets are S -poset isomorphisms sets some limits on dual equivalences between categories of S -posets.

Proposition 4.1 (Cf. Remark 4.5 of [5]). *No category of right S -posets Pos_S can be dually equivalent to a category of left T -posets ${}_T\text{Pos}$.*

Proof. Let $F : \text{Pos}_S \rightarrow {}_T\text{Pos}$ be a dual equivalence, with converse $G : {}_T\text{Pos} \rightarrow \text{Pos}_S$.

Take $\mathbf{1} = \{0\}$ as the subset of $\mathbf{2}$. Consider both these posets as right S -posets with only zero elements, i.e. $1s = 1, 0s = 0$ for all $s \in S$. Then $\mathbf{1} \hookrightarrow \mathbf{2}$ is evidently a convex subobject. Therefore we have $F(\mathbf{2}) \cong F(\mathbf{1})$ in ${}_T\text{Pos}$, since a dual equivalence takes weighted limits to weighted colimits and vice versa as a consequence of it being a special case of enriched adjunction (on this, see [4], pp. 39–40), so it carries convex regular subobjects into coconvex factor objects. But then $\mathbf{2} \cong (GF)(\mathbf{2}) \cong (GF)(\mathbf{1}) \cong \mathbf{1}$ in Pos_S , which is clearly impossible. $\square$

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UNIVERSITY OF TARTU, TARTU, ESTONIA

Email address: ltart@ut.ee