

ENDO-COMPLETELY-REGULAR SPLIT GRAPHS

APIRAT WANICHSONBAT

ABSTRACT. In [Endomorphism-regularity of split graphs, Eur. J. Combin. **22** (2001), 207–216], Weimin Li and Jianfei Chen studied split graphs for which the monoid of all endomorphisms is regular. In this paper, we study which regular endomorphism monoids of split graphs are completely regular and characterize the groups and the monoids of endo-completely-regular split graphs.

1. PRELIMINARIES

All graphs will be finite undirected without loops and multiple edges. If G is a graph, we denote by $V(G)$ (or simply G) and $E(G)$ its vertex set and edge set, respectively. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Moreover, if for any $a, b \in V(H)$, $\{a, b\} \in E(H)$ if and only if $\{a, b\} \in E(G)$, then we call H an *induced subgraph* of G . We denote by K_n a complete graph with n vertices. Let G be a graph and let H be a clique (i.e., a complete subgraph) of G . If H has the maximal order of all the cliques of G , i.e., $|V(A)| \leq |V(H)|$ for any clique A of G , then H is called a *maximal clique* of G . An *independent* set of G is a set of pairwise non-adjacent vertices and a *complete* set is a set which induces a clique. Let H be a subgraph of G and $v \in H$. Denote $N_H(v) := \{x \in H \mid \{x, v\} \in E(H)\}$, call it the *neighborhood* of v in H ; use $N(v)$ for $N_H(v)$ if it is clear which graph H is referred to.

A graph $G(V, E)$ is called a *split graph* if its vertex-set can be partitioned into disjoint (non-empty) sets I and K , i.e., $V = K \cup I$, such that I is an independent set and K is a complete set. (Note: the partition may be not unique, and the clique induced by K is also denoted by K).

Let G and H be graphs. A *homomorphism* $f : G \rightarrow H$ is a vertex-mapping $V(G) \rightarrow V(H)$ which preserves adjacency, i.e., for any $a, b \in V(G)$, $\{a, b\} \in E(G)$ implies that $\{f(a), f(b)\} \in E(H)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H . A homomorphism from G to itself is called an *endomorphism* of G . An isomorphism from a graph G to itself is called an *automorphism* of G . $End(G)$ and $Aut(G)$ denote the set of endomorphisms and automorphisms of graph G , respectively. It is well-known that $End(G)$ is a monoid (i.e., a semigroup with an identity element) and $Aut(G)$ is a group with respect to the composition of mappings. They are often simply called the monoid of G and the group of G , respectively. We denote the semigroup of non-bijective endomorphisms of G by $End'(G)$ instead of $End(G) \setminus Aut(G)$.

Let f be an endomorphism of a graph G . If H is a subgraph of G , by f_H we

2000 *Mathematics Subject Classification*. Primary 05E30.

Key words and phrases. Split graph, completely regular endomorphism.

denote the restriction of f on H ; and $f(H) := \{f(x) | x \in H\}$. Let $G(V, E)$ be a graph. Let $\rho \subseteq V \times V$ be an equivalence relation on V . Denote by $[a]_\rho$ the equivalence class of $a \in V$ under ρ . The graph, denoted by G/ρ , is called the *factor graph* of G under ρ , if $V(G/\rho) = V/\rho$ and $\{[a]_\rho, [b]_\rho\} \in E(G/\rho)$ if and only if there exist $c \in [a]_\rho$ and $d \in [b]_\rho$ such that $\{c, d\} \in E(G)$. Let f be an endomorphism of G . By ρ_f we denote the equivalence relation on $V(G)$ induced by f , i.e., for $a, b \in V(G)$, $(a, b) \in \rho_f$ if $f(a) = f(b)$. The graph G/ρ_f is called the *factor graph by f* .

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. If every element of S is regular, S is called *regular*. A graph G with $\text{End}(G)$ being regular is called *endomorphism-regular*, or simply *endo-regular*. An element a of a semigroup S is called *completely regular* if there exists $x \in S$ such that $axa = a$ and $ax = xa$. A semigroup S is *completely regular* if all its elements are completely regular. A graph G with $\text{End}(G)$ being completely regular is called *endomorphism-completely-regular*, or simply *endo-completely-regular*.

An element z of a semigroup S is a *right (left) zero* if $sz = z$ ($zs = z$) for all $s \in S$; z is a *zero* of S if it is both a right and a left zero of S . A semigroup all of whose elements are right (left) zeros is a *right (left) zero semigroup*. We denote by R_n (L_n) a fixed n element right (left) zero semigroup. For undefined concepts about semigroup in this paper we refer to [2]. The following results quoted from the references will be used later:

Theorem 1.1. (Clifford, [2]) *The following conditions on a semigroup S are equivalent:*

- 1) S is completely regular.
- 2) S is a union of (disjoint) groups.

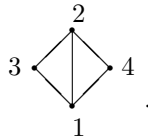
Right groups are the unions of n copies of an arbitrary finite group G and have the form $G \times R_n$, analogously $G \times L_n$ for left groups.

Theorem 1.2. ([3]) *Let $G(V, E)$ be a connected split graph with $V = K \cup I$ and $|K| = n$. Suppose K is a maximal complete set. Then G is endo-regular if and only if there exists $r \in \{1, \dots, n-1\}$ such that $|N(x)| = r$ for all $x \in I$.*

Theorem 1.3. ([3]) *A non-connected split graph G is endo-regular if and only if G exactly consists of a complete graph and several isolated vertices.*

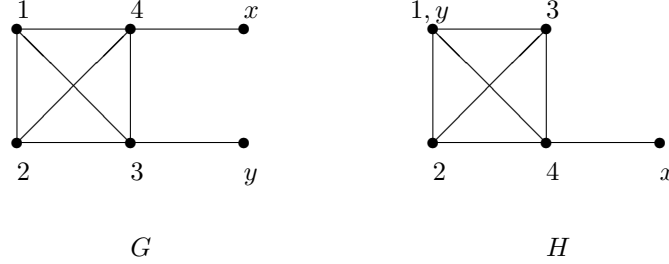
2. COMPLETELY REGULAR ENDOMORPHISMS

From Theorem 1.2 and 1.3, we can conclude that for any split graph G , $\text{End}(G)$ is regular if and only if for all vertices of the independent set, the cardinal numbers of the neighborhoods are equal. From this section a split graph G is always written as $K_n \cup I$ where K_n is a maximal complete subgraph. Note that this may be not unique as the following graph $K_4 \setminus \{e\}$ shows



The first lemma gives some split graphs which are not endo-completely-regular. Before we prove it we give an example which contains the idea of the proof.

Example 2.1. Let G be a graph and H be the factor graph of G under ρ , for which $1\rho y$.



Take the embedding $f = \begin{pmatrix} 1 & 2 & 3 & 4 & x & y \\ 1 & 2 & 4 & 3 & y & 1 \end{pmatrix}$ of H in G . It is clear that f is an endomorphism of G . By Theorem 1.2, there exists an endomorphism g of G such that $f g f = f$. Thus

$$\begin{aligned} (f g f)(x) &= f(x) \\ (f g)(y) &= y. \end{aligned}$$

Consequently, $g(y) = x$, i.e., $(g f)(x) = g(y) = x$. Since $x \notin \text{Im} f$, then $(f g)(x) \neq x = g f(x)$. Hence $\text{End}(G)$ is not completely regular.

Lemma 2.2. Take an endo-regular split graph $G = K_n \cup I$. If $|I| \geq 2$, then $\text{End}(G)$ is not completely regular.

Proof. Let $m := |I| \geq 2$. Suppose that $a_1, a_2 \in I$, $a_1 \neq a_2$ and $V(K_n) = \{1, 2, \dots, n\}$. Consider a mapping f with $f(a_1) = a_2$ and $f(K_n) = K_n$. If G is non-connected, set $f(x) = 1$ for all $x \in I \setminus \{a_1\}$. If G is connected, let the restriction of f to $N(a_1)$ be a bijection to $N(a_2)$ and for any $x \in I \setminus \{a_1\}$, $f(x) \in V(K_n) \setminus \{f(y) | y \in N(x)\}$. It is easy to check that f is an endomorphism of G in both cases. In both cases $a_1 \notin \text{Im} f$. Since G is endo-regular, there exists an endomorphism g such that $f g f = f$. Then

$$(f g)(a_2) = (f g f)(a_1) = f(a_1) = a_2,$$

and thus $g(a_2) = a_1$. Since $(g f)(a_1) = g(a_2) = a_1$ and $a_1 \notin \text{Im} f$, then $(g f)(a_1) \neq (f g)(a_1)$. Hence, we get that $\text{End}(G)$ is not completely regular. $\square$

The next lemma will be used to prove the main theorem in this section. This lemma describes the restriction to the complete subgraph K_n of an endomorphism f of the split graph $K_n \cup I$.

Lemma 2.3. For any split graph $G = K_n \cup I$, let f be an endomorphism on G . If $|N(a)| < n - 1$ for all $a \in I$, then $f(V(K_n)) = V(K_n)$.

Proof. Take $V(K_n) = \{k_1, k_2, \dots, k_n\}$ and f an endomorphism of G . It is clear that for any $i, j \in \{1, 2, \dots, n\}$, $i \neq j$, $f(k_i) \neq f(k_j)$.

Next, we show that for all $i \in \{1, 2, \dots, n\}$, $f(k_i) \in V(K_n)$. Assume that there exists $r \in \{1, 2, \dots, n\}$ such that $f(k_r) = c \in I$. Then $f(K_n) \subseteq N(c) \cup \{c\}$ and

$|f(K_n)| = n$. Thus $|N(c)| = n - 1$, which is a contradiction to the assumption $|N(c)| < n - 1$. $\square$

For any split graph $G = K_n \cup I$, let f be an endomorphism of G . It is obvious that if $|N(a)| = n - 1$ for all $a \in I$, then $|Imf| \geq |V(K_n)|$. Thus by this fact and Lemma 2.3, we can confirm that there are no congruence relation induced by some endomorphism f of any split graph $G = K_n \cup I$ with cardinality less than n .

Theorem 2.4. *For any endo-regular split graph $G = K_n \cup I$, $End(G)$ is completely regular if and only if $|I| = 1$.*

Proof. One implication follows from Lemma 2.2. To prove the reverse implication let $I = \{a\}$ and $V(K_n) = \{1, 2, \dots, n\}$. We can find all congruence relations which we will see in any case.

Case 1. $N(a) = \emptyset$. There are $n + 1$ congruences, namely the identity and n ways of identifying a with one vertex of K_n . Each of these induces an equivalence relation on the corresponding factor graph. Each of these factor graphs can be embedded into G in $n!$ ways according the automorphism group S_n of G . This implies $Aut(G) = S_n$ and $End(G) = \bigcup^{n+1} S_n$, only the multiplication of $End(G)$ needs some thought. By inspection it is clear that $End'(G) = S_n \times R_n$ writing maps on the left.

Case 2. $|N(a)| =: m < n - 1$. The identity which is followed by $(n - m)!m!$ embeddings corresponding to $S_{n-m} \times S_m = Aut(G)$. And $n - m$ non-trivial congruence relations each identifying a with one of its non-neighbors in K_n . Each of these is followed by $n!$ embeddings corresponding to $S_n = Aut(K_n)$. This implies $Aut(G) = S_m \times S_{n-m}$ and $End'(G) = S_n \times R_{n-m}$ as before.

Case 3. $|N(a)| = n - 1$. The identity followed by $2(n - 1)!$ embeddings corresponding to $S_2 \times S_{n-1} = Aut(G)$. There is one non-trivial congruence identifying a with $\{n\} = K_n \setminus N(a)$. This is followed by $2 \cdot n!$ embeddings corresponding to $S_n \times L_2$.

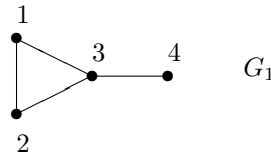
For all cases, we see that $End(G)$ is the union of two groups. Then by Theorem 1.1, we get that $End(G)$ is completely regular. $\square$

Remark 1) We get an equality $|I| = 1$ in the Theorem 2.4, because in this paper an independent set is not empty and K_n is a maximal complete subgraph. However, if an independent set is empty, we get that the split graph is the complete graph K_n . And we also know that $End(K_n)$ is a group. Thus if we allow that an independent set can be the empty set, then we get an inequality $|I| \leq 1$.

2) These split graphs are of the form K_n union K_{m+1} amalgamated at K_m . Here K_0 denotes the empty graph.

Next, we will characterize the monoid of endo-completely-regular split graphs. Before we will consider the monoid of endo-completely-regular split graphs, we want to characterize the semigroup $End'(G)$ of non-bijective endomorphisms.

Example 2.5. a) Considering the graph G_1



we get that $End'(G_1) = \{f_1, f_2, f_3, f_4, f_5, f_6, g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$\begin{aligned} f_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \end{pmatrix}, \\ f_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 3 \end{pmatrix}, \\ g_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 3 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 1 \end{pmatrix}, \\ g_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 3 \end{pmatrix}, g_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 1 \end{pmatrix}, g_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 2 \end{pmatrix}. \end{aligned}$$

Set $F := \{f_i | i = 1, \dots, 6\}$ and $G := \{g_i | i = 1, \dots, 6\}$. We denote the symmetric group $S_3 := \{(1), (12), (13), (23), (123), (132)\}$. It is easy to see that the set F corresponds to S_3 with congruence relation generated by $1\rho 4$, and similarly for G with congruence relation generated by $2\rho 4$. Now we consider multiplication between one endomorphism of F and one from G . For any $f_i \in F$ and $g_j \in G$, we have

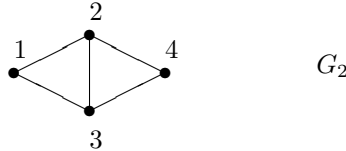
- 1) $f_i|_{K_3} \circ g_j|_{K_3} = f_r|_{K_3} = g_r|_{K_3}$ and $g_j|_{K_3} \circ f_i|_{K_3} = f_s|_{K_3} = g_s|_{K_3}$ for some $r, s \in \{1, \dots, 6\}$,
 - 2) $(f_i \circ g_j)(2) = (f_i \circ g_j)(4)$ and $(g_j \circ f_i)(1) = (g_j \circ f_i)(4)$.
- Then we can conclude that $(f_i \circ g_j) \in G$ and $(g_j \circ f_i) \in F$. And since $f_t|_{K_3} = g_t|_{K_3}$ for all $t \in \{1, \dots, 6\}$, then we have for any $u, v \in \{1, \dots, 6\}$
- 3) $h_u|_{K_3} \circ f_v|_{K_3} = h_u|_{K_3} \circ g_v|_{K_3}$ and $g_v|_{K_3} \circ h_u|_{K_3} = f_v|_{K_3} \circ h_u|_{K_3}$, where $h_u \in \{f_u, g_u\}$.

From these three conditions we can construct the composition table as follows

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4	g_2	g_1	g_5	g_6	g_3	g_4
f_3	f_3	f_4	f_1	f_2	f_6	f_5	g_3	g_4	g_1	g_2	g_6	g_5
f_4	f_4	f_3	f_6	f_5	f_1	f_2	g_4	g_3	g_6	g_5	g_1	g_2
f_5	f_5	f_6	f_2	f_1	f_4	f_3	g_5	g_6	g_2	g_1	g_4	g_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1	g_6	g_5	g_4	g_3	g_2	g_1
g_1	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
g_2	f_2	f_1	f_5	f_6	f_3	f_4	g_2	g_1	g_5	g_6	g_3	g_4
g_3	f_3	f_4	f_1	f_2	f_6	f_5	g_3	g_4	g_1	g_2	g_6	g_5
g_4	f_4	f_3	f_6	f_5	f_1	f_2	g_4	g_3	g_6	g_5	g_1	g_2
g_5	f_5	f_6	f_2	f_1	f_4	f_3	g_5	g_6	g_2	g_1	g_4	g_3
g_6	f_6	f_5	f_4	f_3	f_2	f_1	g_6	g_5	g_4	g_3	g_2	g_1

and we get that $End'(G_1)$ is isomorphic to the right group $S_3 \times R_2$.

b) Considering the graph G_2



we get that $End'(G_2) = \{f_1, f_2, f_3, f_4, f_5, f_6, g_1, g_2, g_3, g_4, g_5, g_6\}$, where

$$f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \end{pmatrix}, f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 1 \end{pmatrix}, f_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 2 \end{pmatrix},$$

$$\begin{aligned} f_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 2 \end{pmatrix}, f_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{pmatrix}, f_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 3 \end{pmatrix}, \\ g_1 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 4 \end{pmatrix}, g_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 4 \end{pmatrix}, g_3 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 2 \end{pmatrix}, \\ g_4 &= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 2 \end{pmatrix}, g_5 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 3 \end{pmatrix}, g_6 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 3 \end{pmatrix}. \end{aligned}$$

Define $F := \{f_i | i = 1, \dots, 6\}$ and $G := \{g_i | i = 1, \dots, 6\}$. Similarly with a), we get F and G isomorphic to S_3 . Since the graph G_2 has two complete subgraphs, then set $K_3 = \{1, 2, 3\}$ and $I = \{4\}$. Now we know that for any $i = 1, \dots, 6$, $Im f_i = N(4) \cup \{1\} =: C$ and $Im g_i = N(4) \cup \{4\} =: D$. Next, we want to consider the multiplication between two elements of F and G . For any $f_i \in F$ and $g_j \in G$, we have $(f_i \circ g_j)(N(4)) \subset C$. And it is easy to check that $(f_i \circ g_j)(1) = (f_i \circ g_j)(4) \in C \setminus (f_i \circ g_j)(N(4))$ and $Im(f_i \circ g_j) = C$. Then we have $(f_i \circ g_j) \in F$. Similarly, we get $(g_j \circ f_i) \in G$. For $r \in \{1, \dots, 6\}$ we observe

- 1) $f_r(a) = g_r(a)$ for all $a \in N(4)$ or
- 2) there exists only one vertex in $N(4)$ such that f_r sends it to 1 and g_r sends it to 4 and for other vertices f_r and g_r send them to the same vertex.

By all conditions above we can conclude for any $h_i \in \{f_i, g_i\}$ and any vertex $a \in N(4)$, $(h_i \circ f_j)(a) = (h_i \circ g_j)(a)$ and we can construct the composition table as follows

$\circ$	f_1	f_2	f_3	f_4	f_5	f_6	g_1	g_2	g_3	g_4	g_5	g_6
f_1	f_1	f_2	f_3	f_4	f_5	f_6	f_1	f_2	f_3	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_4	f_1	f_2	f_6	f_5	f_3	f_4	f_1	f_2	f_6	f_5
f_4	f_4	f_3	f_6	f_5	f_1	f_2	f_4	f_3	f_6	f_5	f_1	f_2
f_5	f_5	f_6	f_2	f_1	f_4	f_3	f_5	f_6	f_2	f_1	f_4	f_3
f_6	f_6	f_5	f_4	f_3	f_2	f_1	f_6	f_5	f_4	f_3	f_2	f_1
g_1	g_1	g_2	g_3	g_4	g_5	g_6	g_1	g_2	g_3	g_4	g_5	g_6
g_2	g_2	g_1	g_5	g_6	g_3	g_4	g_2	g_1	g_5	g_6	g_3	g_4
g_3	g_3	g_4	g_1	g_2	g_6	g_5	g_3	g_4	g_1	g_2	g_6	g_5
g_4	g_4	g_3	g_6	g_5	g_1	g_2	g_4	g_3	g_6	g_5	g_1	g_2
g_5	g_5	g_6	g_2	g_1	g_4	g_3	g_5	g_6	g_2	g_1	g_4	g_3
g_6	g_6	g_5	g_4	g_3	g_2	g_1	g_6	g_5	g_4	g_3	g_2	g_1

and we get that $End'(G_2)$ is isomorphic to the left group $S_3 \times L_2$.

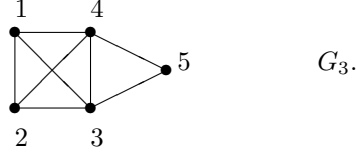
In the same way we get the theorem which gives the structure of $End'(G)$ where G is an endo-completely-regular split graph.

Theorem 2.6. *For any endo-completely-regular split graph $G = K_n \cup \{a\}$ set $|N(a)| := m$. We have:*

- 1) *if $0 \leq m < n - 1$, then $End'(G) = S_n \times R_{n-m}$,*
- 2) *if $m = n - 1$, then $End'(G) = S_n \times L_2$.*

Next we give an example of a group of endo-completely-regular split graph G .

Example 2.7. a) Consider the following graph G_3



We know that $I = \{5\}$ and $N(5) = \{3, 4\}$. Set $C := K_4 \setminus N(5)$. Now we can apply Lemma 2.3 to obtain that

1) all automorphisms of this graph fix 5.

And automorphisms permute only 1 and 2, or 3 and 4. Then we can conclude

2) $f(C) = C$ and $f(N(5)) = N(5)$ for an automorphism f .

By these two conditions we can find all automorphisms as follows

$$id, (12), (34), (12)(34)$$

and thus $Aut(G_3) = S_2 \times S_2$.

b) Consider the graph G_2 in Example 2.5. In the same way as in a), we obtain $Aut(G_2) = S_2 \times S_2$.

From the above example we know that the group of an endo-completely-regular split graph $G = K_n \cup \{a\}$ is the cartesian product of two symmetric groups where the indices depend on the cardinal number of $N(a)$. The next theorem describes the monoids of endo-completely-regular split graphs. Its proof is clear from the preceding examples.

Theorem 2.8. *For any endo-completely-regular split graph $G = K_n \cup \{a\}$, set $|N(a)| := m$. We have:*

- 1) if $m = 0$, then $End(G) = S_n \cup (S_n \times R_n)$,
- 2) if $0 < m < n - 1$, then $End(G) = (S_{n-m} \times S_m) \cup (S_n \times R_{n-m})$,
- 3) if $m = n - 1$, then $End(G) = (S_{n-1} \times S_2) \cup (S_n \times L_2)$.

Remark. Note that from Theorem 2.8 we have in case 1) all endomorphisms are half strong, locally strong endomorphisms are automorphisms, so these graphs have endotype 2, in case 2) all endomorphisms are locally strong, quasi strong endomorphisms are automorphisms, so these graphs have endotype 4, and in case 3) all endomorphisms are strong, so these graphs have endotype 16 (for details see [1]).

Acknowledgement. The author would like to thank Professor Dr. Ulrich Knauer for discussions on this paper.

REFERENCES

- [1] M. Böttcher, U. Knauer, *Endomorphism spectra of graphs*, Discrete Math. **109** (1992), 45–57.
- [2] M. Petrich, N.R. Reilly, *Completely Regular Semigroups*, Wiley Interscience, 1999.
- [3] W. Li, J. Chen, *Endomorphism-regularity of split graphs*, Eur. J. Combin. **22** (2001), 207–216.

Email address: apirat589@yahoo.com