

ON ENDOMORPHISMS OF CLIFFORD SEMIGROUPS

SOMNUEK WORAWISET

ABSTRACT. In this paper we study the endomorphisms of Clifford semigroups, i.e., of strong semilattices of groups. We describe some groups such that their endomorphism monoids are regular and apply some of the results to Clifford semigroups.

1. INTRODUCTION AND PRELIMINARIES

An element a of a semigroup S is called *regular* if there exists an element $x \in S$ satisfying $a = axa$. The semigroup S is called *regular* if all of its elements are regular. An element x of S with $a = axa$ and $x = xax$ is called an *inverse* of a . We denote the set of all inverses of a by $V(a)$.

For a regular semigroup S we define a relation $\leq$ as follows:

$$a \leq b \text{ iff } a = eb = bf \text{ for some } e, f \in E(S) \text{ } (a, b \in S).$$

It is known that this relation is a partial order on regular semigroups [9]. The relation $\leq$ is called the *natural partial order* on S . Several equivalent definitions of this natural partial order on a regular semigroup are known, such results can be seen in [7]. In this paper we use the following of these definitions:

Let S be a semigroup, $a, b \in S$,

$$a \leq b \text{ if there exists } x \in V(a) \text{ such that } ax = bx \text{ and } xa = xb.$$

We note that the natural partial order on a group G is trivial.

Let Y be a semilattice and let $\{G_\alpha\}_{\alpha \in Y}$ be a collection of groups. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\varphi_{\alpha, \beta} : G_\alpha \rightarrow G_\beta$ be a group homomorphism such that $\varphi_{\alpha, \alpha}$ is the identity mapping and if $\alpha > \beta > \gamma$ then $\varphi_{\alpha, \gamma} = \varphi_{\beta, \gamma} \varphi_{\alpha, \beta}$. Let $S = \bigcup_{\alpha \in Y} G_\alpha$ with multiplication

$$a * b = \varphi_{\alpha, \alpha\beta}(a) \varphi_{\beta, \alpha\beta}(b)$$

for $a \in G_\alpha$ and $b \in G_\beta$. The semigroup S is called a *strong semilattice Y of groups G_α* . We call $\varphi_{\xi, \eta}, \xi, \eta \in Y$, the *defining homomorphisms* of S .

An element a of a semigroup S is called *completely regular* if there exists an element $x \in S$ such that $a = axa$ and $ax = xa$. A semigroup S is called *completely regular* if all its elements are completely regular. A semigroup S is called a *Clifford semigroup* if it is completely regular and its idempotents commute with all elements of S ; alternatively, we say that its *idempotents are central* or that they are *in the center*.

Theorem 1.1. [3] *For a semigroup S the following are equivalent:*

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- (i) S is a Clifford semigroup,
- (ii) for every $s \in S$ there exists $x \in S$ such that $sxs = s$ and $sx = xs$,
- (iii) S is a semilattice of groups,
- (iv) S is a strong semilattice of groups.

Let R and S be semigroups and $f : R \rightarrow S$ a mapping, then f is called a *semi-group homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in R$. The set of (semigroup) homomorphisms is denoted by $\text{Hom}(R, S)$ and we use $\text{End}(S) = \text{Hom}(S, S)$.

A semigroup S is said to be an *ordered semigroup* if S is a semigroup with a partial ordering which is compatible i.e, for $a, b \in S$ if $a \leq b$, then $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. We denote an ordered semigroup by $(S, \cdot, \leq)$.

Let $(A, \leq), (B, \leq')$ be ordered semigroups. A mapping $f : A \rightarrow B$ is said to be *order preserving homomorphism* if it is a (semigroup) homomorphism and $x \leq y$ implies that $f(x) \leq' f(y)$ for all $x, y \in A$.

McAlister [6] has shown that the natural order is compatible with multiplication in Clifford semigroups. We shall usually denote this ordering by the symbol $\leq$ and for each $a, b \in S$ we write $a < b$ to mean $a \leq b$ and $a \neq b$.

For any notation and terminology which are not defined in this paper refer to [1], [3] and [9].

2. NATURALLY ORDERED CLIFFORD SEMIGROUP AND SOME PROPERTIES OF ENDOMORPHISMS

First we describe the natural ordering in a Clifford semigroup.

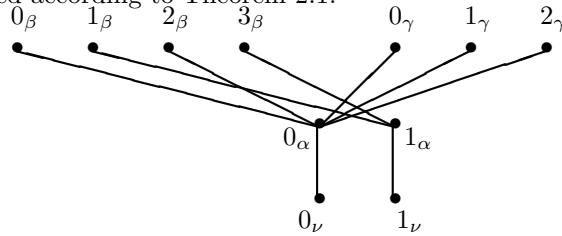
Theorem 2.1. *Let Y be a finite semilattice, $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ . Take $a \in G_\alpha, b \in G_\beta$, $\alpha, \beta \in Y$, then $a \leq b$ if and only if $\alpha \leq \beta$ and $\varphi_{\beta, \alpha}(b) = a$.*

Proof. Let $\varphi_{\beta, \alpha}(b) = a$ and $\alpha \leq \beta$. Since $a^{-1} \in G_\alpha$ and $aa^{-1}a = a$ and $ba^{-1} = \varphi_{\beta, \alpha}(b)a^{-1} = aa^{-1}$ and $a^{-1}b = a^{-1}\varphi_{\beta, \alpha}(b) = a^{-1}a$ we get $a \leq b$.

Conversely, let $a \leq b$. Then there exists $x \in G_\gamma \subseteq S$ for some $\gamma \in Y$ such that $axa = a$, $xa = xb$ and $ax = bx$. So from $axa = a$ we get $\alpha \leq \gamma$ and $ax = bx$ implies $a = axa = bxa$. We have $\alpha \leq \beta$.

From $ax = bx$ it follows that $e_\alpha ae_\gamma = e_\alpha axx^{-1} = e_\alpha bxx^{-1} = e_\alpha be_\gamma$, thus $a = (e_\alpha \varphi_{\beta, \alpha}(b)) \varphi_{\gamma, \alpha}(e_\gamma) = \varphi_{\beta, \alpha}(b)$. $\square$

Example 2.2. Consider the Clifford semigroup $S = \mathbb{Z}_{2_\nu} \cup \mathbb{Z}_{2_\alpha} \cup \mathbb{Z}_{3_\gamma} \cup \mathbb{Z}_{4_\beta}$, with a 4 element semilattice Y and $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ the group with addition modulo $n, n \in \{2, 3, 4\}$, with Hasse diagram as shown below. The defining homomorphisms are as indicated according to Theorem 2.1.



So we have $0_\nu \leq 2_\beta$ since $\varphi_{\beta, \nu}(2_\beta) = 0_\nu$. Thus $0_\nu \leq 0_\xi$ for all $\xi \in Y$, $0_\nu \leq 2_\gamma$, $1_\alpha \leq 1_\beta$ and so on.

Now we study the endomorphisms of S which preserve order, we denote this set by $OEnd(S)$. If for each $\alpha \in Y$, G_α is a group then the identity will be denoted by e_α and the *constant mapping* is denoted by c_{e_α} .

Lemma 2.3. *Let Y be a finite semilattice, $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ and $f \in End(S)$. Then for each $\alpha \in Y$, $f(G_\alpha) \subseteq G_\beta$ for some $\beta \in Y$.*

Proof. Let $x \in G_\alpha$ and $f(x) = a \in G_\beta, \beta \in Y$. Consider x^{-1} , $e_\alpha \in G_\alpha$. Suppose that $f(x^{-1}) = b \in G_\gamma$ and $f(e_\alpha) = e_\delta \in G_\delta, \gamma, \delta \in Y$. Then

$$e_\delta = f(e_\alpha) = f(xx^{-1}) = f(x)f(x^{-1}) = ab.$$

Thus we get $\delta = \beta\gamma \leq \beta$. From $a = f(x) = f(xe_\alpha) = f(x)f(e_\alpha) = ae_\delta$, we get $\beta = \beta\delta \leq \delta$. This implies $\delta = \beta$. From $\beta = \delta = \beta\gamma \leq \gamma$ we have $b = f(x^{-1}) = f(x^{-1}e_\alpha) = f(x^{-1})f(e_\alpha) = be_\beta$, and thus $\gamma = \gamma\beta \leq \beta$. Consequently $\beta = \gamma = \delta$.

Take $y \neq x \in G_\alpha$. If $f(y) = c \in G_\lambda$ for some $\lambda \in Y$, we will show that $\lambda = \beta$. As above, we get $f(y^{-1}) = f(y)^{-1} = c^{-1} \in G_\lambda$. Thus $e_\beta = f(e_\alpha) = f(yy^{-1}) = f(y)f(y^{-1}) = cc^{-1} = e_\lambda$ implies $\beta = \lambda$. Hence $f(G_\alpha) \subseteq G_\beta$. $\square$

For $f \in End(S)$ we get from Lemma 2.3 that for $\alpha \in Y$ one has $f(G_\alpha) \subseteq G_{\alpha'}$, $\alpha' \in Y$. This defines the *induced index mapping* $\underline{f} : Y \rightarrow Y$ with $\underline{f}(\alpha) = \alpha'$ in the obvious way. Thus $f(G_\alpha) \subseteq G_{\underline{f}(\alpha)}$.

We will use the *restriction* $f|_{G_\alpha}$ of f with the usual meaning. For each $f_\alpha \in Hom(G_\alpha, G_{\alpha'})$, we write $f_\alpha(x)$ which implies that x is considered in G_α , but $f(x_\alpha)$ if f is defined on all of S .

We will show in the following theorem that $OEnd(S) = End(S)$ for a Clifford semigroup S .

Theorem 2.4. *Let Y be a finite semilattice, $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ . Then $OEnd(S) = End(S)$.*

Proof. Take $f \in End(S)$, $a_\alpha \in G_\alpha, b_\beta \in G_\beta$ with $a_\alpha \leq b_\beta$. By Theorem 2.1, we get $\varphi_{\beta,\alpha}(b_\beta) = a_\alpha$ and $\alpha \leq \beta$. Suppose that $f(a_\alpha) = x_{\alpha'} \in G_{\alpha'}$ and $f(b_\beta) = y_{\beta'} \in G_{\beta'}$. From $e_\alpha = \varphi_{\beta,\alpha}(e_\beta) = \varphi_{\beta,\alpha}(b_\beta b_\beta^{-1}) = \varphi_{\beta,\alpha}(b_\beta)\varphi_{\beta,\alpha}(b_\beta^{-1}) = a_\alpha b_\beta^{-1}$ we get $e_{\alpha'} = f(e_\alpha) = f(a_\alpha b_\beta^{-1}) = f(a_\alpha)f(b_\beta^{-1}) = (x_{\alpha'})(y_{\beta'})^{-1}$, so that $x_{\alpha'} = y_{\beta'} e_{\alpha'}$. We have $\varphi_{\beta',\alpha'}(y_{\beta'}) = x_{\alpha'}$, therefore $f(a_\alpha) = x_{\alpha'} \leq y_{\beta'} = f(b_\beta)$ by Theorem 2.1. Therefore $f \in OEnd(S)$ and consequently $End(S) = OEnd(S)$. $\square$

3. ENDOMORPHISMS OF CLIFFORD SEMIGROUPS

It has been shown in [2] that if Y is a finite semilattice such that $End(Y)$ is regular, then Y is either a tree or a lattice. Note that Y itself is a Clifford semigroup with one-element groups inside. Now we consider endomorphisms of Clifford semigroups with arbitrary finite groups inside with certain conditions where the semilattice is a finite chain. Later we will study Clifford semigroups with regular monoid.

In Proposition II.2.8 of [8] all homomorphisms of Clifford semigroups are described. In our case we will describe the endomorphism of $S = \bigcup_{\xi \in Y} G_\xi$, strong

semilattice of groups G_ξ with defining homomorphisms being constants or identities in Theorems 3.1 and 3.3 respectively.

Theorem 3.1. *Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ and $\varphi_{\beta,\alpha}(a_\beta) = e_\alpha$ for all $\alpha < \beta \in Y$. Let f be a mapping of S to S , then $f \in \text{End}(S)$ if and only if*

- a) *the induced index mapping $\underline{f} : Y \rightarrow Y$ preserves the ordering on Y ;*
- b) *$f|_{G_\alpha} : G_\alpha \rightarrow G_{\underline{f}(\alpha)}$ is a group homomorphism;*
- c) *For $\alpha' \in \text{Im}(\underline{f})$ set $\underline{f}^{-1}(\alpha') = \{\alpha_0, \alpha_1, \dots\} \subseteq Y$ in the natural order. Then $f(G_{\alpha_i}) = \{e_{\alpha'}\}$ for all $i \geq 1$.*

Proof. Necessity. Let $f \in \text{End}(S)$. By Theorem 2.1, Lemma 2.3 and Theorem 2.4, we get necessity of a) and b).

From the ordering in S we have $e_{\alpha_0} \leq x$ for all $x \in G_{\alpha_i}$, $i \geq 1$. Thus $e_{\alpha'} = f(e_{\alpha_0}) \leq f(x)$. The ordering in $G_{\alpha'}$ is trivial and therefore $f(G_{\alpha_i}) = \{e_{\alpha'}\}$ for all $i \geq 1$.

Sufficiency. Let f be a mapping from S to S satisfying a), b), c). Take $x_\alpha, y_\beta \in S$, $\alpha \leq \beta \in Y$. We have to show that $f \in \text{End}(S)$.

If $\alpha = \beta$, then $f_\alpha = f|_{G_\alpha} : G_\alpha \rightarrow G_{\underline{f}(\alpha)}$ is a group homomorphism and we get $f(x_\alpha y_\beta) = f(x_\alpha)f(y_\beta)$ by b).

Suppose that $\alpha < \beta$. Then $\alpha' = \underline{f}(\alpha) \leq \underline{f}(\beta) = \beta'$ by a). We compute $f(x_\alpha y_\beta) = f(x_\alpha \varphi_{\beta,\alpha}(y_\beta)) = f(x_\alpha e_\alpha) = f(x_\alpha)$ and $f(x_\alpha)f(y_\beta) = f(x_\alpha)\varphi_{\beta',\alpha'}(f(y_\beta))$.

If $\alpha' = \beta'$ we have $f(G_\beta) = \{e_{\alpha'}\}$ by c) and thus $\varphi_{\beta',\alpha'}(f(y_\beta)) = f(y_\beta) = e_{\alpha'}$.

If $\alpha' < \beta'$ we have $\varphi_{\beta',\alpha'}(f(y_\beta)) = e_{\alpha'}$.

So in all cases we have $f(x_\alpha y_\beta) = f(x_\alpha)f(y_\beta)$ and thus $f \in \text{End}(S)$. $\square$

Remark 3.2. From the order preserving property of endomorphisms we get that $\underline{f}^{-1}(\alpha') = \{\alpha_0, \alpha_1, \dots\}$ from Theorem 3.1 is a chain, i.e., $\alpha_{i+1} > \alpha_i$ for every i .

In the next theorem we consider Clifford semigroups where all groups are isomorphic, we say they are equal to G , defining homomorphisms are “identities”.

For these Clifford semigroups we write $x_\alpha \in G_\alpha$ if we consider the element $x \in G$ in the copy G_α .

Theorem 3.3. *Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ and $\varphi_{\beta,\alpha}(a) = a_\alpha$ for all $\alpha \leq \beta \in Y$, $G_\xi = G$ for all $\xi \in Y$ and $a \in G$. Then $f \in \text{End}(S)$ if and only if the following conditions hold*

- a) *the induced index mapping $\underline{f} : Y \rightarrow Y$ preserves the ordering on Y ;*
- b) *$f|_{G_\alpha} : G \rightarrow G$ is a group endomorphism;*
- c) *$f_\alpha(x) = f|_{G_\alpha}(x) = f|_{G_\beta}(x) = f_\beta(x)$ for all $x \in G$ and $\alpha, \beta \in Y$.*

Proof. Necessity. Condition a) comes from Theorem 2.1, Lemma 2.3 and Theorem 2.4, b) since $f \in \text{End}(S)$ and c) follows using that the defining homomorphisms are identities.

Sufficiency. Let f satisfy a), b) and c). We will show that f is a homomorphism. Let $x_\alpha, y_\beta \in S$ with $\alpha \leq \beta$.

Case 1. If $\alpha = \beta$, then $f(x_\alpha y_\alpha) = f(x_\alpha)f(y_\alpha)$ by b).

Case 2. If $\alpha < \beta$, then we get $f(\alpha) \leq f(\beta)$ by a). Then $f(x_\alpha y_\beta) = f(x_\alpha \varphi_{\beta, \alpha}(y_\beta)) = f(x_\alpha y_\alpha) = f(x_\alpha) f(y_\alpha)$ by b) and $f(x_\alpha) f(y_\beta) = f(x_\alpha) f_\beta(y) = f(x_\alpha) f_\alpha(y) = f(x_\alpha) f(y_\alpha)$ by c), i.e., $f(x_\alpha y_\beta) = f(x_\alpha) f(y_\beta)$.

Therefore $f \in \text{End}(S)$. $\square$

Construction 3.4. Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ . Let $\varphi_{\beta, \alpha}(a_\beta) = e_\alpha$ for all $\alpha < \beta \in Y$. Take $\alpha \in Y$ and $f_\alpha \in \text{End}(G_\alpha)$. Define $f : S \rightarrow S$ as follows

$$f(x_\xi) := \begin{cases} f_\alpha(x) & \text{if } \xi = \alpha, x \in G_\xi, \\ e_\xi & \text{if } \xi \neq \alpha, \end{cases}$$

for every $x_\xi \in S$. Then $f \in \text{End}(S)$.

Proof. The statement follows from Theorem 3.1. $\square$

Construction 3.5. Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ . Let $\varphi_{\beta, \alpha}(a) = a_\alpha$ for all $\alpha \leq \beta \in Y$ and $G_\xi = G$ for all $\xi \in Y$. Take $\alpha \in Y$ and $f_\alpha \in \text{End}(G_\alpha)$. Define $f : S \rightarrow S$ as follows

$$f(x_\xi) := \begin{cases} f_\alpha(x) & \text{if } \xi = \alpha, \\ \varphi_{\alpha, \xi}(f_\alpha(x)) & \text{if } \xi < \alpha, \\ y_\xi \in G_\xi \text{ with } \varphi_{\xi, \alpha}(y) = f_\alpha(x) & \text{if } \xi > \alpha, \end{cases}$$

where $x, y \in G$ with $x = x_\xi \in G_\xi$, $y = y_\xi \in G_\xi$. So in fact $y = f_\alpha(x) \in G$ and $\varphi_{\alpha, \xi}(f_\alpha(x)) = f_\alpha(x) \in G$. Then $f \in \text{End}(S)$.

Proof. The statement follows from Theorem 3.3. $\square$

4. REGULAR ENDOMORPHISMS OF GROUPS

Denote by $\text{Hom}(G, H)$ the set of group homomorphisms from a group G to a group H . A map $f \in \text{Hom}(G, H)$ is called *regular* if there exists $g \in \text{Hom}(H, G)$ such that $f g f = f$. A subset of $\text{Hom}(G, H)$ is called *regular* if each of its elements is regular.

A subgroup A of a group G is called a *direct summand* of G if there is a subgroup B of G such that $A \cap B = \{e\}$ and $A + B = G$, where e is the identity in G . We write $G = A \oplus B$.

Remark 4.1. If $(G, +)$ and $(H, *)$ are abelian, i.e., commutative groups, then the set $\text{Hom}(G, H)$ of all group homomorphisms from G to H is itself an abelian group: the sum $h + k$ of two homomorphisms is defined by

$$(h + k)(x) = h(x) * k(x)$$

for all x in G . The commutativity of H is needed to prove that $h + k$ is again a group homomorphism. The addition of homomorphisms is compatible with the composition of homomorphisms in the following sense: if $f \in \text{Hom}(K, G)$, $h, k \in \text{Hom}(G, H)$, and $g \in \text{Hom}(H, L)$, then

$$(h + k) \circ f = (h \circ f) + (k \circ f) \quad \text{and} \quad g \circ (h + k) = (g \circ h) + (g \circ k).$$

This shows that the set $End(G)$ of all endomorphisms of an abelian group forms a ring, the endomorphism ring of G .

Example 4.2. We can see that if a group G is not abelian, then the set $(End(G), +)$ is not closed. Take the symmetric group S_3 , and $f, g \in End(S_3)$ with

$$f = \begin{pmatrix} (1) & (123) & (132) & (23) & (13) & (12) \\ (1) & (1) & (1) & (23) & (23) & (23) \end{pmatrix},$$

$$g = \begin{pmatrix} (1) & (123) & (132) & (23) & (13) & (12) \\ (1) & (1) & (1) & (13) & (13) & (13) \end{pmatrix}.$$

Then $(f + g)((12)(12)) = (f + g)((1)) = (1) \neq (123) = ((23)(13))((23)(13)) = (f + g((12)))(f + g((12)))$. Therefore $f + g \notin End(S_3)$.

From [5] we have that for an abelian group G the endomorphism ring of G is regular if and only if images and kernels of all endomorphisms of G are direct summands of G . The next theorem will generalize this to the set of homomorphisms $Hom(G, H)$ where G and H are abelian groups. The proof is the same as in the case where G and H are modules over a ring R given by [4].

Theorem 4.3. *Let G and H be abelian groups. Let $f \in Hom(G, H)$. Then f is regular if and only if $Ker(f)$ is a direct summand of G and $Im(f)$ is a direct summand of H .*

Proof. Assume that f is regular, then there exists $g \in Hom(H, G)$ such that $f g f = f$. Then we have $Im(f) = Im(f g f) \subseteq Im(f g) \subseteq Im(f)$ and $Ker(f) \subseteq Ker(g f) \subseteq Ker(f g f) = Ker(f)$. Therefore $Im(f) = Im(f g)$ and $Ker(f) = Ker(g f)$. Thus it is straightforward to verify that $G = Ker(f) \oplus Im(g f)$ and $H = Im(f) \oplus Im(1_H - f g)$ where 1_H is identity map on H and $(1_H - f g)(x) = 1_H(x) + f g(x^{-1})$ for all $x \in H$.

Conversely, suppose that $G = Ker(f) \oplus G_1$ and $H = Im(f) \oplus H_1$ such that G_1 is a subgroup of G and H_1 is a subgroup of H . Then $\bar{f} : G_1 \rightarrow Im(f)$ by $x \mapsto f(x)$ is an isomorphism. Let $g : H \rightarrow G$ be the map with $g = c_{e_G}$ on H_1 and $g = \bar{f}^{-1}$ on $Im(f)$. Then $f g f = f$. $\square$

Example 4.4. Consider the set of homomorphisms $Hom(\mathbb{Z}_6, \mathbb{Z}_4)$. Take $f \in Hom(\mathbb{Z}_6, \mathbb{Z}_4)$ as follows

$x \in \mathbb{Z}_6$	0	1	2	3	4	5
$f(x) \in \mathbb{Z}_4$	0	2	0	2	0	2

It can be seen that $Im(f) = \{0, 2\} \subseteq \mathbb{Z}_4$, $\{0, 2\}$ is not a direct summand of $\mathbb{Z}_4$ and $Ker(f) = \{0, 2, 4\} \subseteq \mathbb{Z}_6$ and $\{0, 2, 4\}$ is a direct summand of $\mathbb{Z}_6$. Then we have f is not regular by Theorem 4.3.

5. APPLICATIONS TO CLIFFORD SEMIGROUPS

We specialize Theorem 4.3 to the monoid of endomorphisms of S where $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ . In this case only one direction stays valid.

Theorem 5.1. *Let Y be a finite semilattice and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of abelian groups G_ξ . Let $f \in \text{End}(S)$ be regular then for each $\alpha \in Y$, $\text{Im}(f|_{G_\alpha})$ is a direct summand of G_β and $\text{Ker}(f|_{G_\alpha})$ is a direct summand of G_α with $f(G_\alpha) \subseteq G_\beta$ for some $\beta \in Y$.*

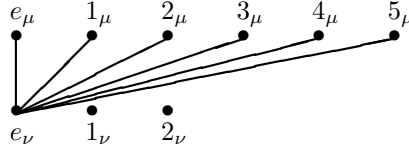
Proof. Assume that $fgf = f$ where $g \in \text{End}(S)$. For each $\alpha \in Y$, $f(G_\alpha) \subseteq G_\beta$ for some $\beta \in Y$ by Lemma 2.3 and by Theorem 4.3 we have $\text{Im}(f|_{G_\alpha})$ is a direct summand of G_β and $\text{Ker}(f|_{G_\alpha})$ is a direct summand of G_α . $\square$

On the other hand, even if $f \in \text{End}(S)$ is such that $\text{Im}(f|_{G_\xi})$, $\xi \in Y$, is a direct summand of G_γ , $f(G_\xi) \subseteq G_\gamma$, $\gamma \in Y$, and $\text{Ker}(f|_{G_\xi})$ is a direct summand of G_ξ , it may happen that f is not regular. We will show this in the next example.

Example 5.2. Consider the Clifford semigroup $S = \mathbb{Z}_{3\nu} \cup \mathbb{Z}_{6\mu}$, Y a 2 element chain, the defining homomorphisms being zero maps and

$$f = \begin{pmatrix} 0_\nu & 1_\nu & 2_\nu & e_\mu & 1_\mu & 2_\mu & 3_\mu & 4_\mu & 5_\mu \\ 0_\mu & 2_\mu & 4_\mu & e_\mu & e_\mu & e_\mu & e_\mu & e_\mu & e_\mu \end{pmatrix}.$$

Then $\text{Im}(f|_{G_\nu}) = \{0_\mu, 2_\mu, 4_\mu\}$ is a direct summand of G_μ and $\text{Ker}(f|_{G_\nu}) = \{e_\nu\}$ is a direct summand of G_ν and so on, but f has no inverse element.



Theorem 5.3. *Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ and $\varphi_{\beta,\alpha}(a_\beta) = e_\alpha$ for all $\alpha < \beta \in Y$. If $\text{End}(S)$ is regular then $\text{End}(G_\xi)$ is regular for all $\xi \in Y$.*

Proof. Let $\text{End}(S)$ be regular. Take $f_\alpha \in \text{End}(G_\alpha)$. Then we construct f as in Construction 3.4. Thus $f \in \text{End}(S)$. By assumption there exists $g \in \text{End}(S)$ such that $fgf = f$. Consider $x_\alpha \in G_\alpha$, $f_\alpha(x_\alpha) \in G_\alpha$ and $g_\alpha(G_\alpha) \subseteq G_\beta$ for some $\beta \in Y$ by Lemma 2.3. By Construction 3.4 we see that $\beta = \alpha$, i.e., $g_\alpha(G_\alpha) \subseteq G_\alpha$. This implies $f_\alpha g_\alpha f_\alpha = f_\alpha$ and $g_\alpha \in \text{End}(G_\alpha)$. Therefore $\text{End}(G_\alpha)$ is regular. $\square$

We will proceed now to the case where the defining homomorphisms are identity maps, and S consists of copies of G .

Theorem 5.4. *Let Y be a finite chain and $S = \bigcup_{\xi \in Y} G_\xi$ a strong semilattice of groups G_ξ and $\varphi_{\beta,\alpha}(a) = a_\alpha$ for all $\alpha \leq \beta \in Y$, $G_\xi = G$ for all $\xi \in Y$ and $a \in G$. If $\text{End}(S)$ is regular then $\text{End}(G)$ is regular.*

Proof. Let $\text{End}(S)$ be regular. We will show that $\text{End}(G) = \text{End}(G_\xi)$ is regular for all $\xi \in Y$.

Take an arbitrary endomorphism of G , $f_\alpha \in \text{End}(G) = \text{End}(G_\alpha)$.

Construct f with f_α as in Construction 3.5 and $f_\alpha(x) \in G_\alpha$ for all $\alpha \in Y$. Note that x is an element in G , considered in G_α . Then $f \in \text{End}(S)$. Regularity of f gives $g \in \text{End}(S)$ with $fgf = f$.

For $x \in G_\alpha$ we get $f(x) \in G_\alpha$ and thus $g_\alpha(G_\alpha) \subseteq G_\beta = G$ for some $\beta \in Y$ by Lemma 2.3. This implies $f_\beta g_\alpha f_\alpha = f_\alpha$ where $f_\beta = f_\alpha \in \text{End}(G)$ by Construction 3.5 and $g_\alpha \in \text{End}(G)$. So $\text{End}(G)$ is regular. $\square$

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INSTITUT FÜR MATHEMATIK, CARL VON OSSIETZKY UNIVERSITÄT, D-26111 OLDENBURG, GERMANY

Email address: nuek_w@yahoo.com