

CLIFFORD SEMIGROUPS WITH GENUS ZERO

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*Dedicated to Professor Ulrich Knauer and Professor Mati Kilp
on the occasion of their 65th birthdays*

ABSTRACT. We classify Clifford semigroups with genus zero whose semilattice contains two elements and groups are finite.

1. PRELIMINARIES

It is known that each group can be defined in terms of generators and relations, and that corresponding to such a presentation there is a unique graph, called Cayley graph of the presentation. A “drawing” of this graph gives a “picture” of the group from which certain properties of the group can be determined.

A graph is said to be *imbedded* into a surface M if it is “drawn” on M such that edges do not intersect outside vertices. A graph is said to be *planar* if it can be embedded into the plane. By the *genus* of a graph G we mean the minimum genus among all surfaces in which G can be embedded. So if G is planar then the genus of G is zero.

Finite planar groups have been catalogued by Maschke in [5]. On the basis of Maschke Theorem, in this work we classify Clifford semigroups with genus zero whose semilattice contains two elements and groups are finite.

As usual every semigroup presentation is associated with a Cayley color graph: the vertices correspond to the elements of the semigroup; next, imagine the generators from a minimal generating set of the semigroup to be associated with distinct colors. If vertices v_1 and v_2 correspond to semigroup elements s_1 and s_2 respectively, then there is a directed edge (of the color of generator e) from v_1 to v_2 if and only if $s_1 e = s_2$. In this work we consider the pseudograph obtained from the Cayley color graph by suppressing all edge directions and all edge colors, that is, an undirected, uncolored Cayley graph. It is clear that in passing from the Cayley color graph to the corresponding pseudograph only structural properties but not topological considerations are lost.

Let Y be a semilattice, $S = \bigcup_{\alpha \in Y} G_\alpha$, where $(G_\alpha)_{\alpha \in Y}$ is a family of disjoint groups indexed by Y . Suppose that for all $\alpha \geq \beta$ in Y there exists a group homomorphism $f_{\alpha,\beta} : G_\alpha \rightarrow G_\beta$ such that

- (1) $f_{\alpha,\alpha} = id_{G_\alpha}$ for all $\alpha \in Y$,
- (2) for all $\alpha \geq \beta \geq \gamma$ in Y , $f_{\beta,\gamma} f_{\alpha,\beta} = f_{\alpha,\gamma}$.

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Define multiplication in $S = \bigcup_{\alpha \in Y} G_\alpha$ by

$$xy = f_{\alpha, \alpha \wedge \beta}(x) f_{\beta, \alpha \wedge \beta}(y)$$

for $x \in G_\alpha, y \in G_\beta$. Then S is said to be a strong semilattice Y of groups $G_\alpha, \alpha \in Y$, with defining homomorphisms $f_{\alpha, \beta}, \alpha \geq \beta$; S defined in this manner is a semigroup, indeed a Clifford semigroup. Conversely, any Clifford semigroup may be constructed in this way (see [2], [6]).

Here we use the concept of a Clifford semigroup S as a strong semilattice Y of groups G_α, G_β with Y being a 2-element chain. We write $S = G_\alpha \bigcup G_\beta$, where $\alpha > \beta$, with the defining group homomorphism $f_{\alpha, \beta}: G_\alpha \rightarrow G_\beta$.

The reader is referred to [3], [7], [1] and [6] for the terminology and notions which are not given in this paper. We need the following results.

Result 1.1. (Maschke [5]) *A finite group Γ is planar if and only if $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_1 = \mathbb{Z}_1$ or $\mathbb{Z}_2$ and $\Gamma_2 = \mathbb{Z}_n, D_n, S_4, A_4$ or A_5 .*

Result 1.2. (Kuratowski [4]) *A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.*

2. THE WAY TO THE MAIN THEOREM

We first consider such Clifford semigroups S whose groups are of type $\mathbb{Z}_n$. For convenience we write the element from $\mathbb{Z}_n$ by x instead of $\bar{x}$ for $x \in \mathbb{N}$, that is, $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$. A comment about generating sets is in order. Obviously a generating set of S can be obtained by simply joining the generating sets of its groups. In computing the genus of a group, changing the minimal generating set can not increase the genus ([1]). For the situation of the defining homomorphism $f_{\alpha, \beta}: \mathbb{Z}_m \rightarrow \mathbb{Z}_n$, if $f_{\alpha, \beta}$ is not the zero mapping and $f_{\alpha, \beta}(1_\alpha)$ is a generating element then without loss of generality, we always may take $f_{\alpha, \beta}(1_\alpha) = 1_\beta$. The two planar groups produce two graphs, the outer graph for the upper group and the inner graph for the lower group, their edges will be solid lines. Multiplication in S which is defined with the defining homomorphism will provide two kinds of subgraphs: the subgraph of the lower group with the generating set from the upper group as connection set, the new-inner graph, edges are dashed lines, which are possible new lines in the inner graph; and the subgraph of the upper group with the generating set from the lower group as connection set, the in-between graph, edges are also dashed lines, which go from the outer graph to the inner graph. For a group G with generating set K , we write $G = \langle K \rangle$.

Proposition 2.1. *Let $S = G_\alpha \bigcup G_\beta$ be a Clifford semigroup with $\alpha > \beta$, $G_\alpha = \mathbb{Z}_m$, $G_\beta = \mathbb{Z}_n$. Then S is planar if and only if one of the following cases holds:*

- (1) $f_{\alpha, \beta}$ is the zero mapping,
- (2) $m = n$ and $f_{\alpha, \beta}$ is bijective,
- (3) $n = 2m$ and $f_{\alpha, \beta}$ is injective.

Proof. Sufficiency. If $f_{\alpha, \beta}$ is the zero mapping then each vertex from $\mathbb{Z}_n$ gets a loop and each vertex from $\mathbb{Z}_m$ gets an edge towards the generating element of $\mathbb{Z}_n$. Clearly the new edges don't destroy the planarity.

For the second case, suppose $\mathbb{Z}_m = \langle 1_\alpha \rangle$ and $f_{\alpha, \beta}(1_\alpha) = x_\beta$. Since $f_{\alpha, \beta}$ is bijective we may assume $\mathbb{Z}_n = \langle x_\beta \rangle$. It turns out that $\mathbb{Z}_n$ itself does not add

new edges and the added edges between $\mathbb{Z}_m$ and $\mathbb{Z}_n$ are parallel, i.e., have no intersections.

If $n = 2m$ and $f_{\alpha,\beta}$ is injective then without loss of generality set $f_{\alpha,\beta}(1_\alpha) = 2_\beta$. Let $\{1_\alpha, 1_\beta\}$ be the generating set of S . The outer graph is an m -gon and the inner graph is a $2m$ -gon. By suitably choosing the order of vertices the new-inner graph is the m -gon $0_\beta, 2_\beta, \dots, (2m-2)_\beta$, together with the m -gon $1_\beta, 3_\beta, \dots, (2m-1)_\beta$. The in-between graph has the edges $(0_\beta, 1_\beta), (1_\beta, 2_\beta), \dots, ((2m-2)_\beta, (2m-1)_\beta), ((2m-1)_\beta, 0_\beta)$. As an example, we give the pictures of the Cayley graph $\text{Cay}(\mathbb{Z}_3 \cup \mathbb{Z}_6, \{1_\alpha, 1_\beta\})$ (Figure 1) and the respective planar graph (Figure 2).

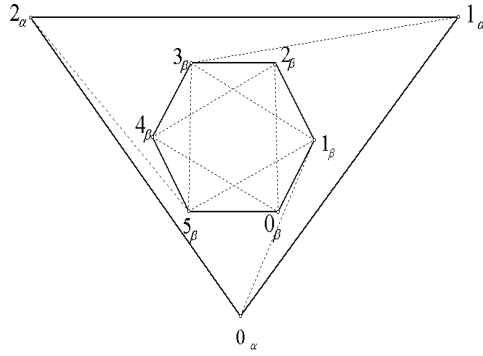


Figure 1

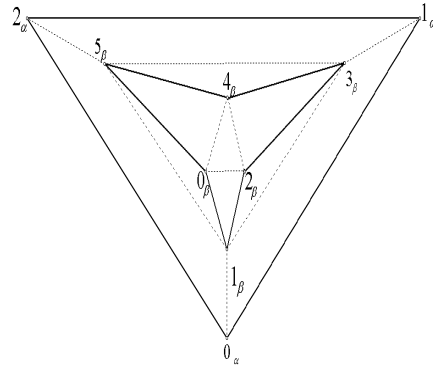


Figure 2

Necessity. We need to show S is not planar if it does not fulfill one of the three conditions. Firstly we consider the case that $m < n$ and set $f_{\alpha,\beta}(1_\alpha) = x \in \mathbb{Z}_n$. We claim that the subgraph with the vertices of $\mathbb{Z}_n$ in $\text{Cay}(S, \{1_\alpha, 1_\beta\})$ is non-planar. Since $(x, n) \neq 1$ as $m < n$, and $x = 2$ gives that $n = 2m$ which is the condition of (3), it follows that $x \neq 2$ and $x|n$.

Now $\mathbb{Z}_n$ contains x copies of $\frac{n}{x}$ -gons whose vertices in turn are $\{0, x, \dots, n-x\}$, $\{1, x+1, \dots, n-x+1\}$, $\dots$, $\{x-1, 2x-1, \dots, n-1\}$, the new-inner graph. We will consider the subdivision whose vertex set is $\{0, x-1, x+1\} \cup \{1, x, n-1\}$ of the graph.

Together with the inner graph 0 is adjacent to $1, x, n-1$; 1 to $0, x+1, 2$; and finally $n-1$ to $n-2, x-1, 0$. Therefore, 1 being adjacent to $x-1$ is obtained by a finite sequence of subdivisions of edges $(1, 2), (2, 3), \dots, (x-2, x-1)$, and $x+1$ is adjacent to $n-1$ by a finite sequence of subdivisions of edges $(x+1, 2x+1), (2x+1, 3x+1), \dots, (n-x-1, n-1)$. Consequently we get a subdivision of the inner graph together with the new-inner graph which is $K_{3,3}$ and by Result 1.2, S is not planar.

The simplest example is $m = 2, n = 6, f_{\alpha,\beta}(1_\alpha) = 3$, and we get the subgraph (Figure 3) with the vertices of $\mathbb{Z}_6$ of the respective Cayley graph (the inner and the new-inner graph) which is $K_{3,3}$ (Figure 4). Thus the whole Cayley graph is not planar.

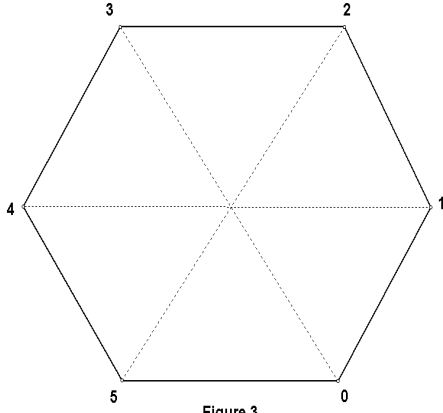


Figure 3

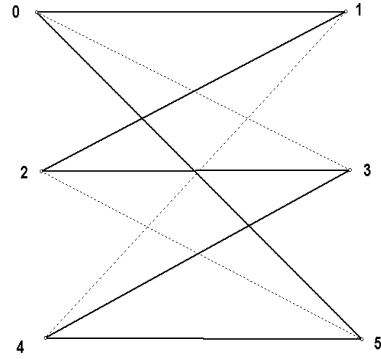


Figure 4

Next assume that $m \geq n$, $1_\alpha, 1_\beta$ are generating elements of $\mathbb{Z}_m$ and $\mathbb{Z}_n$ respectively, and $f_{\alpha,\beta}(1_\alpha) = x_\beta$. Then 0_α is adjacent to $1_\alpha, 1_\beta$, and $(n-1)_\alpha$; 1_α is adjacent to $(x+1)_\beta$ and 2_α ; 1_β is adjacent to $(x+1)_\beta$ and 2_α by a finite sequence of subdivisions of edges $(2_\alpha, 3_\alpha), (3_\alpha, 4_\alpha), \dots, (k_\alpha, 1_\beta)$, where $k = \frac{[x,n]}{x}$, and $[x,n]$ is the least common multiple of x and n ; $(n-1)_\alpha$ is adjacent to $(x+1)_\beta$ by a finite sequence of subdivisions of edges $((n-1)_\alpha, (n-2)_\alpha), ((n-2)_\alpha, (n-3)_\alpha), \dots, ((k+1)_\alpha, (x+1)_\beta)$ in which $k = \frac{[x,n]}{x}$, and adjacent to 2_α by a finite sequence of subdivisions of edges $(2_\alpha, (2x+1)_\beta), ((2x+1)_\beta, (3x+1)_\beta), \dots, (((n-2)x+1)_\beta, ((n-1)x+1)_\beta), (((n-1)x+1)_\beta, (n-1)_\alpha)$. Again we get a subdivision of $\text{Cay}(S, \{1_\alpha, 1_\beta\})$ which is $K_{3,3}$. We show the pictures of the Cayley graph $\mathbb{Z}_9 \cup \mathbb{Z}_3$ with the defining group homomorphism $f_{\alpha,\beta}(1_\alpha) = 2_\beta$.

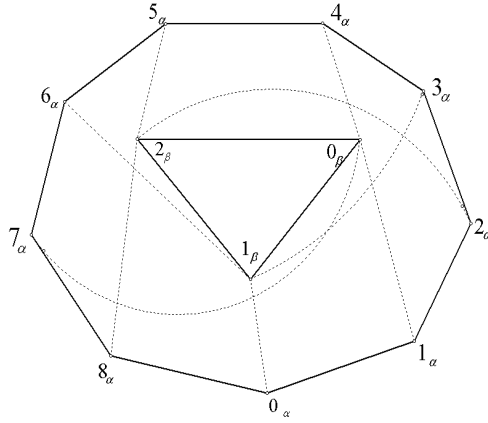


Figure 5

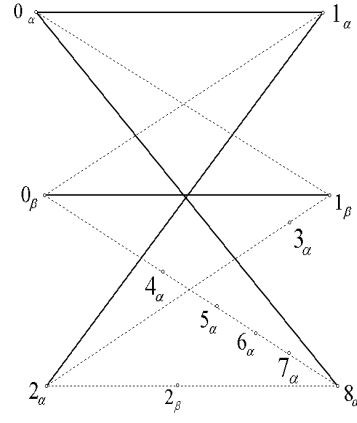


Figure 6

□

In the following we replace the upper group of S in Proposition 2.1 by arbitrary planar group. We will use the notion $G_\alpha \xrightarrow{f} G_\beta$ to denote the groups and the respective defining group homomorphism $f_{\alpha,\beta}$. By $G_\alpha \xrightarrow{(f,f')} G_\beta$ we mean the

corresponding restricted mappings of $f_{\alpha,\beta}$ on each component of G_α if G_α has two generators. As usual, 0 denotes the zero mapping, id denotes the identity mapping.

Note that $\mathbb{Z}_k \times \mathbb{Z}_m$ and $\mathbb{Z}_{km}$ are isomorphic groups when k and m are coprime, i.e., $(k, m) = 1$.

Proposition 2.2. *Let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with $\alpha > \beta$, G_α is any finite planar group, $G_\beta = \mathbb{Z}_n$. Then S is planar if and only if one of the following conditions holds:*

- (1) $G_\alpha = \mathbb{Z}_m$ and
 - (a) $\mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_n$, or
 - (b) $m = n$ with $\mathbb{Z}_m \xrightarrow{\text{bijective}} \mathbb{Z}_m$, or
 - (c) $n = 2m$ with $\mathbb{Z}_m \xrightarrow{\text{injective}} \mathbb{Z}_{2m}$,
- (2) $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_m$ with $(2, m) = 1$ and
 - (a) $\mathbb{Z}_{2m} \xrightarrow{0} \mathbb{Z}_n$, or
 - (b) $n = 2m$ with $\mathbb{Z}_{2m} \xrightarrow{\text{bijective}} \mathbb{Z}_{2m}$, or
 - (c) $n = 4m$ with $\mathbb{Z}_{2m} \xrightarrow{\text{injective}} \mathbb{Z}_{4m}$,
- (3) $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_2$ and
 - (a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_n$, or
 - (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(id,0)} \mathbb{Z}_2$, or
 - (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(injective,0)} \mathbb{Z}_4$.

Proof. Sufficiency. If $G_\alpha = \mathbb{Z}_m$ or $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_m$ with $(2, m) = 1$ then the planarity of S is as in Proposition 2.1 by the fact that $\mathbb{Z}_2 \times \mathbb{Z}_m \cong \mathbb{Z}_{2m}$. Otherwise if $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_m$ with $m = 2$ then the Cayley graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ inside $Cay((\mathbb{Z}_2 \times \mathbb{Z}_2) \cup \mathbb{Z}_n, \{1_1, 1_2\})$ is a square and clearly S is planar when $f_{\alpha,\beta}$ is the zero mapping, where $1_1, 1_2$ are generators of $\mathbb{Z}_2 \times \mathbb{Z}_2$. For the case of (3), if $n = 2$ with $f_{\alpha,\beta}(1_1) = f_{\alpha,\beta}(1_2)$ then the Cayley graph is isomorphic to $K_{3,3}$.

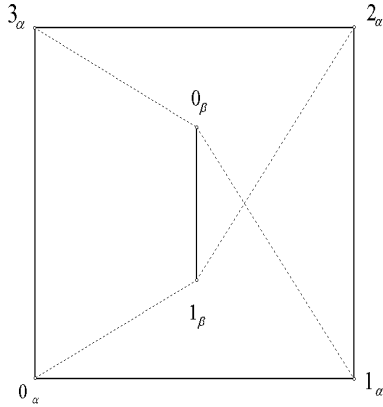


Figure 7

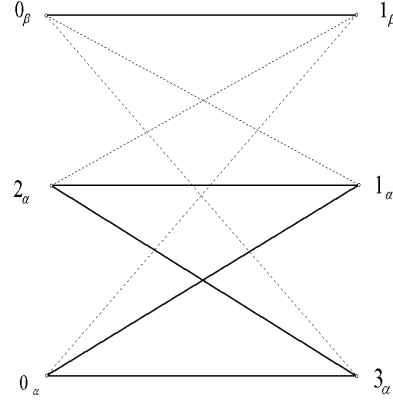


Figure 8

If $n = 4$ with $f_{\alpha,\beta}(1_1) = f_{\alpha,\beta}(1_2)$ then between the two planar groups four new edges are added in which the identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $1_1 1_2$ are adjacent to a common vertex of $\mathbb{Z}_4$, and $1_1, 1_2$ are adjacent to another common vertex of $\mathbb{Z}_4$, this leads to

a crossing and correspondingly to a subdivision of the graph of S which is $K_{3,3}$. It is easy to check that S is planar if $f_{\alpha,\beta}(1_1) \neq f_{\alpha,\beta}(1_2)$.

Necessity. Suppose that $G_\alpha \neq \mathbb{Z}_2 \times \mathbb{Z}_m$ is a planar group from Result 1.1 whose minimal generating set contains at least two generators, and $f_{\alpha,\beta}$ is the zero mapping. Choose a generator s , identity e , vertices s', s'' being adjacent to s from G_α , and $1 \in \mathbb{Z}_m$. Obviously every vertex from G_α is adjacent to 1 and we have edges $(e, s), (s, s'), (s, s'')$. Note that each vertex of G_α provides at least two different paths, which are back against the generators, i.e., do not pass through the generators, to any other vertex in G_α . Therefore we get an (s', e) -path, an (s'', e) -path, an (s', s'') -path, which are pairwise different, and hence a subdivision of the graph of S which is K_5 . For example we show the pictures of G_α being A_4 , where the generators of A_4 are a, b and set $c = bbaba$. It is sufficient to consider only the vertex 1 in $\mathbb{Z}_m$ to see the graph is not planar.

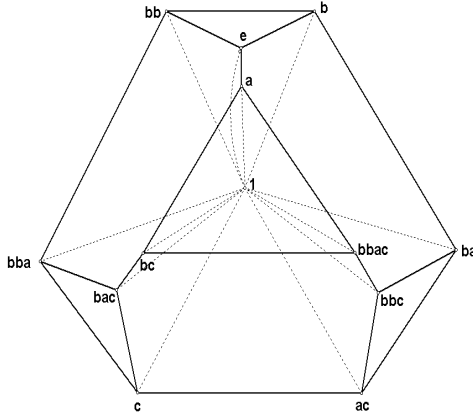


Figure 9

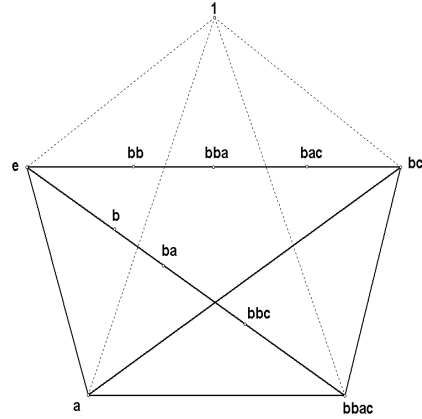


Figure 10

Similar discussions apply to the situation of $f_{\alpha,\beta}$ being non-zero. We will get a subdivision of the graph of S which is either K_5 or $K_{3,3}$. Again we choose $G_\alpha = A_4$ and let $G_\beta = \mathbb{Z}_2$ with $f_{\alpha,\beta}(a) = 1, f_{\alpha,\beta}(b) = 1$. Figure 11 and Figure 12 show the corresponding Cayley graph of S and its subdivision which is $K_{3,3}$.

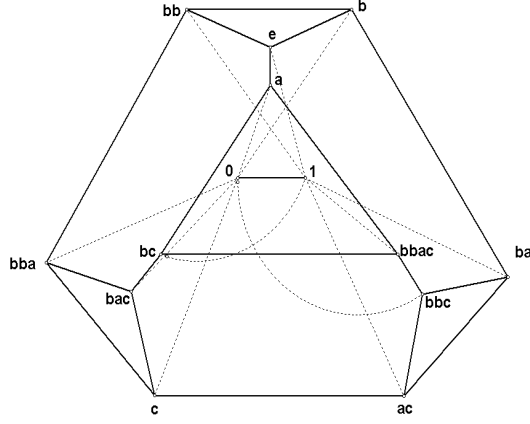


Figure 11

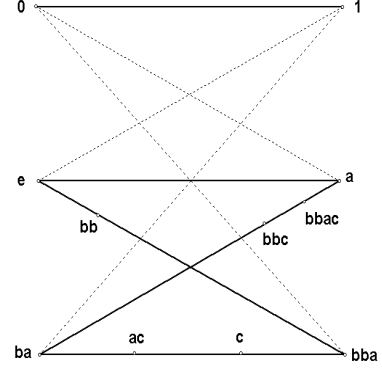


Figure 12

□

Next we turn to Clifford semigroups S with D_n as lower group.

Proposition 2.3. *Let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with $\alpha > \beta$, G_α is any finite planar group, $G_\beta = D_n$. Then S is planar if and only if one of the following conditions holds:*

- (1) $n = 1$,
 - (a) $G_\alpha = \mathbb{Z}_m$ with $\mathbb{Z}_m \xrightarrow{0} D_1 = \mathbb{Z}_2$, or
 - (b) $G_\alpha = \mathbb{Z}_2$ with $\mathbb{Z}_2 \xrightarrow{\text{bijective}} D_1 = \mathbb{Z}_2$, or
 - (c) $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{0} D_1 = \mathbb{Z}_2$, or
 - (d) $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(id,0)} D_1 = \mathbb{Z}_2$,
- (2) $n = 2$, $G_\alpha = \{0\}$ or $G_\alpha = \mathbb{Z}_2$,
- (3) $n > 2$,
 - (a) $G_\alpha = \{0\}$, or
 - (b) $G_\alpha = \mathbb{Z}_2$ with $\mathbb{Z}_2 \xrightarrow{0} D_n$.

Proof. If $n = 1$ then D_1 is the cyclic group $\mathbb{Z}_2$. The statement follows from Proposition 2.1 and Proposition 2.2.

If $n = 2$ then $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$. Since every non-identity of $\mathbb{Z}_2 \times \mathbb{Z}_2$ belongs to a minimal generating set, it turns out by suitably choosing generating elements of $\mathbb{Z}_2 \times \mathbb{Z}_2$, that S is planar whatever $f_{\alpha,\beta}$ from $\mathbb{Z}_2$ to $\mathbb{Z}_2 \times \mathbb{Z}_2$ is used. Considering $|G_\alpha| > 2$, if $f_{\alpha,\beta}$ is the zero mapping then one will get a subdivision of S being K_5 whose vertices are: two generators from $\mathbb{Z}_2 \times \mathbb{Z}_2$, three vertices from the G_α which constitute a cycle by adding some extra vertices. Suppose $f_{\alpha,\beta}$ is not zero mapping. We may assume that $f_{\alpha,\beta}(e') = 1_1$, where e' is an element in a minimal generating set of G_α , $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{e, 1_1, 1_2, 1_1 1_2\} = \langle \{1_1, 1_2\} \rangle$ with e being the identity. Then we get $K_{3,3}$ whose vertex set is $\{0, e, 1_1 1_2\} \cup \{e', 1_1, 1_2\}$, where 0 is the identity of G_α .

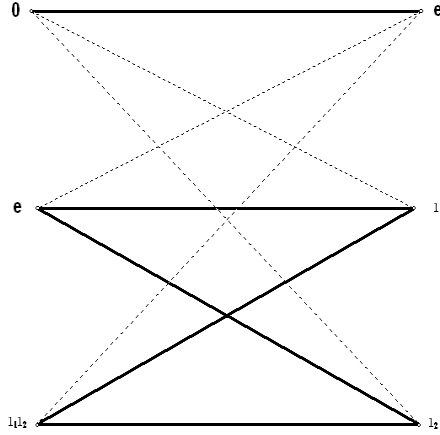


Figure 13

For the case $n > 2$, assume the generating elements of D_n are r and t . Then $D_n = \{e, r, \dots, r^{n-1}, t, rt, \dots, r^{n-1}t\}$. If $G_\alpha = \{0\}$ or $G_\alpha = \mathbb{Z}_2$ with $f_{\alpha,\beta}$ being the zero mapping then the added new edges are between G_α and r, t which provide a planar drawing in the face containing r, t . Otherwise assume $f_{\alpha,\beta}(e') = r^k$, $0 < k < n$, e' is a generator of G_α . Then we have disjoint vertex sets $\{0, r^k t, r^{k+1}t\}$, $\{e', r, t\}$ and edges between them are $(0, e'), (0, r), (0, t), (e', r^k t), (e', r^{k+1}t)$, where 0 is the identity of G_α . Also we have paths: $(t, r^{n-1}t, \dots, r^k t)$, $(t, e, r^{n-1}, \dots, r^{k+1})$, $(r^k t, r^{k-1}t, \dots, rt, r)$, $(r^{k+1}, r^k, \dots, r)$. This gives a subdivision of S which is $K_{3,3}$. $\square$

Proposition 2.3 and Proposition 2.2 imply the following corollary with $\mathbb{Z}_2 \times \mathbb{Z}_n$ being the lower group of S .

Corollary 2.4. *Let $S = G_\alpha \cup G_\beta$ with $\alpha > \beta$, G_α is any finite planar group, $G_\beta = \mathbb{Z}_2 \times \mathbb{Z}_n$, be a Clifford semigroup. Then S is planar if and only if one of the following conditions holds:*

- (1) $G_\alpha = \{0\}$,
- (2) $n = 1$,
 - (a) $G_\alpha = \mathbb{Z}_m$ with $\mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_2$, or
 - (b) $G_\alpha = \mathbb{Z}_2$ with $\mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2$, or
 - (c) $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_2$ with $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(id,0)} \mathbb{Z}_2$,
- (3) $n = 2$, $G_\alpha = \mathbb{Z}_2$,
- (4) $n > 2$, $G_\alpha = \mathbb{Z}_2$ with $\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \times \mathbb{Z}_n$,
- (5) $n > 2$, $(n, 2) = 1$,
 - (a) $G_\alpha = \mathbb{Z}_m$ with $\mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_{2n}$, or
 - (b) $G_\alpha = \mathbb{Z}_{2n}$ with $\mathbb{Z}_{2n} \xrightarrow{bijective} \mathbb{Z}_{2n}$, or
 - (c) $G_\alpha = \mathbb{Z}_n$ with $\mathbb{Z}_n \xrightarrow{injective} \mathbb{Z}_{2n}$.

With respect to A_4, A_5, S_4 , and $\mathbb{Z}_2 \times A_4$ with minimal generating sets containing two elements being the lower groups of S , S is planar only if the upper group has

one element or two elements with the defining group homomorphism being the zero mapping. Indeed, in this case the added new edges will provide either one more quadrangle whose vertices are 0, the generating elements and identity from the upper group or one more quadrangle and two more triangles whose vertices are 0,1, the generating elements and the identity from the upper group. For the situation of $G_\beta = A_4$ we have the two planar graphs (Figure 14 and Figure 15).

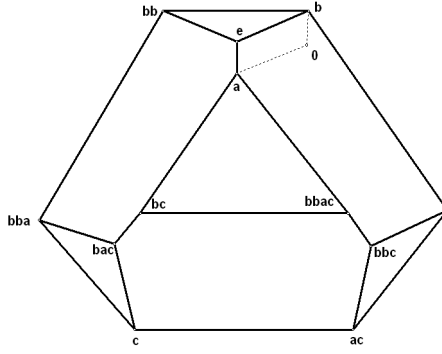


Figure 14

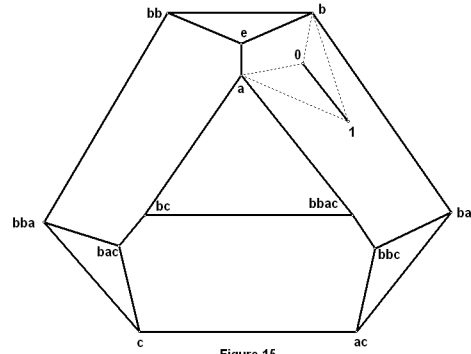


Figure 15

As for the other planar groups with three elements as minimal generating sets being the lower group of S , S is not planar even if the upper group of S has only one element. For instance, if $G_\beta = \mathbb{Z}_2 \times D_2$ then there is a subdivision of $\text{Cay}(S, \{a, b, c\})$, where a, b, c are generators of $\mathbb{Z}_2 \times D_2$, being $K_{3,3}$ as the following diagrams show (Figure 16 and Figure 17).

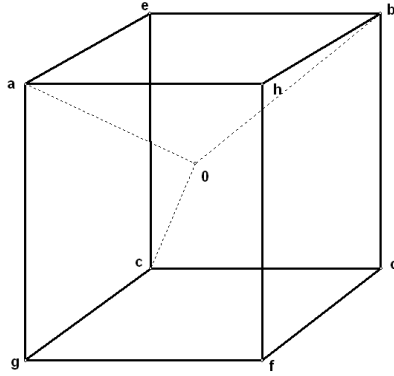


Figure 16

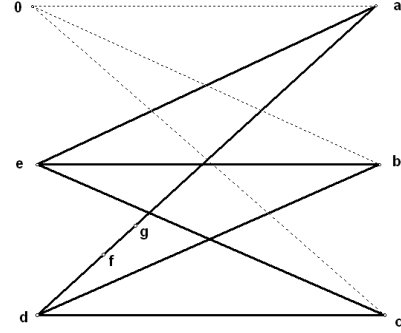


Figure 17

Therefore finally we achieve the classification of planar Clifford semigroups as follows.

Theorem 2.5. *Let $S = G_\alpha \cup G_\beta$ be a Clifford semigroup with $\alpha > \beta$. Then S is planar if and only if one of the following cases holds:*

Case 1: $G_\alpha = \mathbb{Z}_m$, $G_\beta = \mathbb{Z}_n$ with

- (a) $\mathbb{Z}_m \xrightarrow{0} \mathbb{Z}_n$, or
- (b) $m = n$, $\mathbb{Z}_m \xrightarrow{\text{bijective}} \mathbb{Z}_m$, or
- (c) $n = 2m$, $\mathbb{Z}_m \xrightarrow{\text{injective}} \mathbb{Z}_{2m}$.

Case 2: $G_\alpha = \mathbb{Z}_2 \times \mathbb{Z}_2$ with

- (a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_n$, or
- (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(id,0)} \mathbb{Z}_2$, or
- (c) $\mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{(injective,0)} \mathbb{Z}_4$.

Case 3: $G_\alpha = \mathbb{Z}_2$, $G_\beta = \mathbb{Z}_2 \times \mathbb{Z}_2$ or D_2 .

Case 4: $G_\beta = \mathbb{Z}_n$, $\mathbb{Z}_2 \times \mathbb{Z}_{2n}$, D_n , ($n > 2$), A_4 , A_5 , S_4 , or $\mathbb{Z}_2 \times A_4$, and

- (a) $G_\alpha = \{0\}$, or
- (b) $G_\alpha = \mathbb{Z}_2$ with $f_{\alpha,\beta}$ being the zero mapping.

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