TOPOLOGICAL BRANDT λ-EXTENSIONS OF TOPOLOGICAL SEMIGROUPS

Abstract

If a topological-algebraic object which is included in another object, contains in it as a closed subspace, than it is called *H-closed*. Striving to solve the problem of finding of the criterion of the H-closedness and the absolute H-closedness in the category of topological semigroups there was constructed a topologically-algebraic extension (namely, the *topological Brandt λ-extension of topological semigroups*) which preserve both H-closedness and an absolutely H-closedness in the class of topological inverse semigroups. For every infinite cardinal λ, semigroup topologies on Brandt λ-extensions which preserve H-closedness and an absolute H-closedness are constructed. An example of a non H-closed topological inverse semigroup S in the class of topological inverse semigroups such that for any cardinal $\lambda \geq 2$ there exists an absolute H-closed topological Brandt λ–extension of the semigroup S in the class of topological semigroups is constructed. The sufficient conditions on topological Brandt extensions for preserving the (absolute) H-closedness will be given.

As a consequence of the obtained results, the structure of compact 0-simple topological inverse semigroups, structure of compact and countably compact primitive topological inverse semigroups and structure of pseudocompact completely 0-simple topological inverse semigroups will be described.

Using the construction of topological Brandt λ-extensions of topological semigroups, an example of the countable absolutely H-closed 0-dimensional metrizable inverse topological semigroup S with an absolutely H-closed ideal I such that the Rees quotient-semigroup S/I is not a topological semigroup is constructed.

Plan

1. H-closedness and absolute H-closedness problems.
2. Definitions of Brandt extension and topological Brandt extension.
3. (Absolute) H-closedness of Brandt extension in the class of topological inverse semigroups (proof).
4. (Absolute) H-closedness of Brandt extension in the class of topological semigroups: semigroup topologies on Brandt extensions which preserve H-closedness and an absolute H-closedness (constructions).
5. Sufficient conditions on topological Brandt extensions for preserving the (absolute) H-closedness.
6. Application of Brandt extension to describing next classes of topological semigrups:
   - compact 0-simple topological inverse semigroups;
   - compact and countably compact primitive topological inverse semigroups;
   - pseudocompact completely 0-simple topological inverse semigroups.
7. Example of the absolutely H-closed topological semigroup S with an absolutely H-closed ideal I such that the Rees quotient-semigroup S/I is not a topological semigroup.
8. Open problems.

1. H-closedness and absolute H-closedness problems.

If a topological-algebraic object which is included in another object, contains in it as a closed subspace, than it is called *H-closed*.

Let $\mathcal{S}$ be a class of topological semigroups.

A topological semigroup is called *H–closed in $\mathcal{S}$* if it is a closed subsemigroup of any topological semigroup from class $\mathcal{S}$ which contains it as subsemigroup.

A topological semigroup is called *absolutely H–closed in $\mathcal{S}$* if any continuous homomorphic image of it into any topological semigroup from class $\mathcal{S}$ is H–closed in $\mathcal{S}$.

- ***Alexandroff, Urysoh**, 1923*: criterion of *H*–closedness of Hausdorff **topological spaces** [Eng, p.333, 3.12.5 (1-4)]: *each open cover $\{U_s\}_{s\in S}$ of space X contains a finite subfamily $\{U_{s_1},...,U_{s_k}\}$ such that, space could be represented as $X=\overline{U_{s_1}}\cup...\cup\overline{U_{s_k}}$ union of closures of that subfamily*;
- ***Rajkov**, 1946*: a criterion of H–closedness of **topological groups** (*topological group is H-closed if and only if it is Raikov complete, i.e. it is complete with respect to the two-sided uniformity* [Гуран,Зарічний];
- ***Stepp**, 70-ies of XX*: investigated H-closedness of **topological semigroups**; established a criterion of *H*–closedness of discrete semilattices (*arbitrary maximal chain should be finite*); proved that any locally compact topological semigroup is a dense subsemigroup of some *H*-closed topological semigroup;
- ***Ravsky**, 2003:* found sufficient conditions for the commutative topological group to be *H*-closed in the class of paratopological groups (*paratopological group (Bourbaki) is a group with semigroup topology, that is a set with algebraic group structure, and group operation is continuous but inversion not; For example: 1) real line with arrow topology – algebraic group, not topological group, but topological semigroup; 2) any topological group*).

In 1940 ***Katetov*** shown that *H*-closed topological spaces are preserved by continuous functions (continuous image of *H*-closed topological space is *H*-closed space), that is arbitrary *H*-closed topological space is absolutely *H*-closed.

A topological semigroup $S \in \mathcal{S}$ is called *absolutely H-closed* in the class $\mathcal{S}$, if any continuous homomorphic image of *S* into $T \in \mathcal{S}$ is *H*-closed in $\mathcal{S}$.

There exist *H*-closed non-absolutely *H*-closed objects both in the category of topological groups, in category of topological inverse semigroups and in category of topological semigroups.

For example, in the class of topological groups and in the class of topological inverse semigroups it is additive group of integers with discrete topology $(\mathbb{Z},+,discr)$. It is *H*-closed by Raikov criterion, that is Raikov complete (all locally compact groups are Raikov complete). But it is not absolutely H-closed ("irrational torus winding"; consider homomorphism $\mathbb{Z}\to e^{\pi i\sqrt{2}x}\subset\mathbb{C}$, $e^{ix}=\cos x+i\sin x$, its image in the compact $\{z:z\in\mathbb{C},|z|=1\}$ is not closed. None of two points coincide.)

A question of when a topological group is absolutely *H*-closed is not resolved completely.

- ***Dikranjan, Uspenskij**, 1998*: showed that absolutely *H*-closedness in the class of topological groups is preserved by Cartesian products and closed central subgroups (*it means, if subgroup of absolutely H-closed group is central and closed, then it is absolutely H-closed; central group is the set which is subset of a center – elements which commute with all others*).

- ***Stepp*** established a criterion of absolute *H*–closedness of discrete semilattices (*the same as for H-closedness; under homomorphism number of maximal chain could not increase*) and posed a problem "whether arbitrary *H*-closed topological semilattice is absolutely *H*-closed?" It is open question until now.
- ***Gutik, Repovš***, *2008*: proved the criteria of *H*-closedness of linearly ordered semilattices [three conditions, SF08] and they shown that every linearly *H*-closed topological semilattice is absolutely *H*-closed, thus solving Stepp problem for a partial case.

Since category of topological semigroups contain all categories of topological associative algebras (algebra with one associative operation, other are trivial), the question on criterion of *H*-closedness of topological semigroups is very complicated. Still there are no *H*-closedness and absolute *H*-closedness criteria for topological semigroups. Thereby, a ***question on preservation of H-closedness and absolute H-closedness by topologic-algebraic extensions of topological semigroups*** is of current interest.

Topological Brandt λ-extension in the class of topological inverse semigroups, and some semigroup topologisations of Brandt λ-extensions proved to be such exactly.

2. Definitions of Brandt extension and topological Brandt extension.

Definition. Let *S* be a semigroup and I_λ be a set of cardinality λ.

On the set $B_\lambda(S) = I_\lambda \times S^1 \times I_\lambda \cup \{0\}$ we define the semigroup operation "$\cdot$" as follows:

$$(\alpha, a, \beta) \cdot (\gamma, b, \delta) = \begin{cases} (\alpha, ab, \delta), & if\ \beta = \gamma \\ 0, & if\ \beta \neq \gamma \end{cases}$$

and $(\alpha, a, \beta) \cdot 0 = 0 \cdot (\alpha, a, \beta) = 0 \cdot 0 = 0$ for $\alpha, \beta, \gamma, \delta \in I_\lambda$, $a, b \in S^1$. The semigroup $B_\lambda(S)$ is called ***the Brandt λ-extension of the semigroup*** *S* (extension means that it should contain *S* as algebraical subsemigroup and *S* should be topological subsemigroup of that extension; $s \mapsto (\alpha, s, \alpha)$ - algebraical isomorphism, in direct sum topology $V(s) \mapsto (\alpha, V(s), \alpha)$ - continuous embedding, because pre-image is open).

How it could be imagined: as cartezian product $I_\lambda \times I_\lambda$ with a copy of *S* linked to every non-zero point and zero adjoined.

Obviously $B_\lambda(S)$ is the Rees matrix semigroup $\mathcal{M}^0[S^1; I_\lambda, I_\lambda, M]$, where *M* is the $I_\lambda \times I_\lambda$-identity matrix. If the semigroup S is trivial (i.e. S contains only one element), then $B_\lambda(S)$ is the semigroup of matrix units.

Completely 0-simple inverse semigroup is isomorphic to Brandt groupoid, that is being a Brandt λ-extension of some group. In the 80-ies of XX Howie proposed a construction of embedding of semigroups into nilpotent-generated semigroups of nilpotend index *2*. In 1999 Gutik generalized Howie construction for arbitrary cardinal λ≥2 and topologized this construction. The obtained semigroup was called the Brandt λ-extension of the semigroup *S*. Previously in literature this construction was known as *Brandt semigroup* for case S is a group.

Further, if $A \subseteq S^1$ then we shall denote $A_{\alpha\beta} = \{(\alpha, s, \beta) \in B_\lambda(S) \mid s \in A\}$ for $\alpha, \beta \in I_\lambda$ and $S_{\alpha\beta} = \{(\alpha, x, \beta) \in B_\lambda(S) \mid x \in S^1\}$, and by $\mathcal{S}$ we denote some class of topological semigroups.

Definition. Let λ be a cardinal ≥2, and $(S,\tau)\in\mathcal{S}$. Let τ_B be a topology on $B_\lambda(S)$ such that:

1) $(B_\lambda(S),\tau_B)\in\mathcal{S}$,

2) $\tau_B\,|_{S_{\alpha\alpha}}=\tau$ for some $\alpha\in I_\lambda$.

Then $(B_\lambda(S),\tau_B)$ is called ***a topological Brandt λ-extension*** *of* (S,τ) *in* $\mathcal{S}$. If $\mathcal{S}$ coincides with the class of all topological semigroups, then $(B_\lambda(S),\tau_B)$ is called a topological Brandt λ-extension of (S,τ) (such that τ_B induce on $S_{\alpha\alpha}$ topology τ, because $S_{\alpha\alpha}$ is a copy of S; why on $S_{\alpha\alpha}$, to avoid $(\alpha,t,\beta)(\alpha,s,\beta)=0$.)

Let *S* be a topological semigroup. We define topology on $B_\lambda(S)$ in next way. Let $\mathcal{B}_S(s)$ be a base of topology of semigroup S in the point $s\in S$. Obviously, the conditions *(BP1)-(BP3)* hold for family $\mathcal{B}_{B_\lambda(S)}(\alpha,s,\beta)=\{(\alpha,U(s),\beta):U(s)\in\mathcal{B}_S(s)\}$, and hence it determines a base of topology in point $(\alpha,s,\beta)\in B_\lambda(S)\setminus\{0\}$.

It is obvious that the family

$$\mathcal{B}_{B_\lambda(S)}(\alpha,s,\beta)=\{\mathcal{B}_{B_\lambda(S)}(\alpha,s,\beta):s\in S^1,\alpha,\beta\in I_\lambda\}\bigcup\{0\},$$

where 0 is a zero of a semigroup $B_\lambda(S)$, is a base of topology τ_B on the semigroup $B_\lambda(S)$. It is easily verified that $(B_\lambda(S),\tau_B)$ is a topological semigroup, and if *S* be a topological inverse semigroup, then $(B_\lambda(S),\tau_B)$ be a topological inverse semigroup too.

Topological space $B_\lambda(S)$ is homeomorphic to the space $(\prod_{\alpha\in I_\lambda\times I_\lambda}S^1)\coprod\{0\}$, that is why topology τ_B on $B_\lambda(S)$ is called *direct sum topology on semigroup* $B_\lambda(S)$. It should be noted that in this case $(B_\lambda(S),\tau_B)$ is topological inverse semigroup if and only if *S* be a topological inverse semigroup (on B_λ it is discrete topology, but there exist many others).

It is obvious that Brandt λ-extension of arbitrary topological (inverse) semigroup *S* with direct sum topology is a topological Brandt λ-extension of semigroup *S*. That is why topological Brandt λ-extension exists for any topological semigroup *S*.

3. (Absolute) H-closedness of λ-Brandt extension in the class of topological inverse semigroups (proof)

I will prove that a topological inverse semigroup *S* is (absolutely) *H*-closed in the class of topological inverse semigroups if and only if for every cardinal $\lambda\geq 2$ arbitrary topological Brandt λ-extension of the semigroup *S* is (absolutely) *H*-closed semigroup in the class of topological inverse semigroups.

Lemma 1. Let *T* be a dense subsemigroup of a topological semigroup *S* and *0* be the zero of *T*. Then *0* is the zero of *S*.

Proof. Let *0* be a zero of *T*. Suppose that there exists $a\in S\setminus T$ such that $0\cdot a=b\neq 0$. Then for every open neighbourhood $U(b)\not\ni 0$ in *S* there exists an open neighbourhood $V(a)\not\ni 0$ in *S* such that $0\cdot V(a)\subseteq U(b)$. But $|\,V(a)\bigcap T\,|\geq\omega$, and hence $0\in 0\cdot V(a)\subseteq U(b)$, a contradiction with the choice of $U(b)$. Therefore $0\cdot a=0$ for all $a\in S$. The proof of the equality $a\cdot 0=0$ is similar. ◄

Proposition 1. Let λ be an arbitrary cardinal, λ≥2 and $B_\lambda(S)$ be a subsemigroup of topological semigroup *T*. If the set $S_{\alpha\beta}$ is closed in *T* for some $\alpha,\beta\in I_\lambda$, then $S_{\gamma\delta}$ is a closed subset in *T* for all $\gamma,\delta\in I_\lambda$.

Proof. Define the maps $\varphi:T\to T$ and $\psi:T\to T$ for arbitrary $\alpha,\beta,\gamma,\delta\in I_\lambda$ by the formulae: $\varphi(x)=(\alpha,1,\gamma)\cdot x\cdot(\delta,1,\beta)$ and $\psi(x)=(\gamma,1,\alpha)\cdot x\cdot(\beta,1,\delta)$. The maps φ and ψ are continuous as compositions of translations in the topological semigroup *T*. Thus $A=\varphi^{-1}(S_{\alpha\beta})$ is a closed subspace in *T*. Obviously, $S_{\gamma\delta}\subseteq A$. Let $f=\psi\circ\varphi$. Then the map $f_A=f|_A$: $A\to S_{\gamma\delta}$ is a retraction, and the set $S_{\gamma\delta}$ is a retract of topological space *A*. Thus, $S_{\gamma\delta}$ is a closed subspace in topological semigroup *T*. ◄

Proposition 2. Let *A* be a subset of the semigroup $B_\lambda(S)$ such that *A* cross at least two sets $S_{\alpha\beta}$ (non-singleton). Then $0\in A\cdot A$.

Proof. Let $A\cap S_{\alpha_1\beta_1}\neq\varnothing$ and $A\cap S_{\alpha_2\beta_2}\neq\varnothing$ for some $\alpha_1,\alpha_2,\beta_1,\beta_2\in I_\lambda$.

If $\alpha_1\neq\beta_1$, or $\alpha_2\neq\beta_2$, then $0\in A\cdot A$ ($0=(\alpha_1,a,\beta_1)(\alpha_1,b,\beta_1)\in A\cdot A$, or $0=(\alpha_2,a,\beta_2)(\alpha_2,b,\beta_2)\in A\cdot A$).

If $\alpha_1=\beta_1$ and $\alpha_2=\beta_2$, then $\alpha_1\neq\alpha_2$, and $0\in A\cdot A$ ($0=(\alpha_1,a,\alpha_1)(\alpha_2,b,\alpha_2)\in A\cdot A$). ◄

Lemma 2. Let $B_\lambda(S)$ be a dense subsemigroup of topological semigroup *T* and the set $S_{\alpha\beta}$ is closed in *T* for some $\alpha,\beta\in I_\lambda$. Then $a\cdot a=0$ for all $a\in T\setminus B_\lambda(S)$.

Proof. Let $a\cdot a=b\neq 0$ for some $a\in T\setminus B_\lambda(S)$. Then for any open neighbourhood $U(b)$ of element $b\in T$ such that $0\notin U(b)$, there exists an open neighbourhood $V(a)$ of element $a\in T$ such that $0\notin V(a)$ and $V(a)\cdot V(a)\subseteq U(b)$. By proposition 1, $S_{\alpha\beta}$ is a closed subset in *T*, for arbitrary $\alpha,\beta\in I_\lambda$. Since $B_\lambda(S)$ is dense subsemigroup of *T*, arbitrary open neighbourhood of element a cross the semigroups $B_\lambda(S)$. If $V(a)$ cross finite number of sets $S_{\alpha\beta}$, $\alpha,\beta\in I_\lambda$, then from the closedness of $S_{\alpha\beta}$ follows that $V(a)$ contains in some set $S_{\alpha_0\beta_0}$, $\alpha_0,\beta_0\in I_\lambda$, which contradicts with $a\in T\setminus B_\lambda(S)$. The obtained contradiction implies the statement of the theorem. ◄

Proposition 3. Let *A* and *B* be disjunctive subsets of the semigroup $B_\lambda(S)$ each of which cross at least two sets $S_{\alpha\beta}$, where $\alpha,\beta\in I_\lambda$. Then either $0\in A\cdot B$, or $0\in B\cdot A$.

Proof. (*idea: neighborhood indices are distinct and multiplication gives zero*)

Fix elements $\alpha_1,\beta_1,\alpha_2,\beta_2,\alpha_3,\beta_3,\alpha_4,\beta_4\in I_\lambda$ such that $A\cap S_{\alpha_1\beta_1}\neq\varnothing$, $A\cap S_{\alpha_2\beta_2}\neq\varnothing$, $B\cap S_{\alpha_3\beta_3}\neq\varnothing$ and $B\cap S_{\alpha_4\beta_4}\neq\varnothing$.

Since $S_{\alpha_1\beta_1}$ and $S_{\alpha_2\beta_2}$ are disjunctive subsets of semigroup $B_\lambda(S)$, then either $\alpha_1\neq\alpha_2$, or $\beta_1\neq\beta_2$ (*sufficient to them to be distinct*).

Thus, if $\alpha_1\neq\alpha_2$, then either $\alpha_1\neq\beta_3$, or $\alpha_2\neq\beta_3$ (if not then $\alpha_1=\beta_3=\alpha_2$), and $0\in B\cdot A$ (if $\alpha_1\neq\beta_3$, $0=(\alpha_3,a,\beta_3)(\alpha_1,b,\beta_1)\in B\cdot A$; if $\alpha_2\neq\beta_3$, $0=(\alpha_3,a,\beta_3)(\alpha_2,b,\beta_2)\in B\cdot A$).

If $\beta_1\neq\beta_2$, then either $\beta_1\neq\alpha_3$, or $\beta_2\neq\alpha_3$ (if not then $\beta_1=\alpha_3=\beta_2$), and, $0\in A\cdot B$ (if $\beta_1\neq\alpha_3$, $0=(\alpha_1,b,\beta_1)(\alpha_3,a,\beta_3)\in A\cdot B$; if $\beta_2\neq\alpha_3$, $0=(\alpha_2,b,\beta_2)(\alpha_3,a,\beta_3)\in A\cdot B$).◄

Theorem 1. Let $\lambda \geq 2$, $B_\lambda(S)$ be inverse subsemigroup of topological inverse semigroup *T* and set $S_{\alpha\beta}$ be closed in *T* for some $\alpha, \beta \in I_\lambda$. Then $B_\lambda(S)$ is a closed subsemigroup in *T*.

Proof. In case $2 \leq \lambda < \omega$ it follows from proposition 1.

Let $\lambda \geq \omega$ and $cl_T(B_\lambda(S)) = G$. By proposition 1.2.1 [EbSeld], G is a topological inverse semigroup.

Let $b \in G \setminus B_\lambda(S)$. Then by lemma 2, $b, b^{-1} \in G \setminus E(G)$. Since by lemma 1, 0 is the zero of the topological semigroup G, then $b \cdot b^{-1} \neq 0$ and $b^{-1} \cdot b \neq 0$. Otherwise, if $b \cdot b^{-1} = 0$, or $b^{-1} \cdot b = 0$, then $b = b \cdot b^{-1} \cdot b = (b \cdot b^{-1}) \cdot b = 0 \cdot b = 0$, or $b = b \cdot b^{-1} \cdot b = b \cdot (b^{-1} \cdot b) = b \cdot 0 = 0$, a contradiction with $b \in G \setminus B_\lambda(S)$.

Thus there exist $e, f \in E(G) = E(B_\lambda(S))$ such, that $b \cdot b^{-1} = e$ and $b^{-1} \cdot b = f$. Let $e \neq f$ and $W(e)$, $W(f)$ be disjunctive open neighborhoods of idempotents $e, f \in T$, such that $0 \notin W(e)$ and $0 \notin W(f)$. Then there exist disjunctive open neighborhoods $U(b)$ and $U(b^{-1})$ in *T* such that

$$0 \notin U(b), \quad 0 \notin U(b^{-1}), \quad U(b) \cdot U(b^{-1}) \subseteq W(e), \quad U(b^{-1}) \cdot U(b) \subseteq W(f).$$

By proposition 1, $S_{\alpha\beta}$ is a closed subset of topological semigroup *T* for all $\alpha, \beta \in I_\lambda$, thus neighborhoods $U(b)$ and $U(b^{-1})$ cross infinite number of sets $S_{\gamma\delta}$, where $\gamma, \delta \in I_\lambda$. Then by proposition 3 one of next conditions should hold: either $0 \in U(b) \cdot U(b^{-1}) \subseteq W(e)$, or $0 \in U(b^{-1}) \cdot U(b) \subseteq W(f)$, a contradiction with $0 \notin W(e)$, or $0 \notin W(f)$, respectively.

If $e = f$, the proof of the statement is similar.

The obtained contradiction implies the statement of the theorem. ◀

Proposition 4. If *S* is *H*-closed topological semigroup (in class $\mathcal{S}$), then S^1 is *H*-closed topological semigroup (in class $\mathcal{S}$). (obvious - as union of two closed sets)

Theorem 1 and proposition 4 imply:

Theorem 2. For every cardinal $\lambda \geq 2$ any topological Brandt λ-extension of an *H*-closed topological inverse semigroup in the class of topological inverse semigroups is *H*-closed semigroup in the class of topological inverse semigroups.

Let $\mathcal{TIS}$ be a class of topological inverse sesmigroups, and $\mathcal{TS}$ be a class of topological semigroups.

Theorem 3. Let *S* be a topological inverse semigroup . Then the following conditions are equivalent:

1) *S* is an *H*-closed semigroup in the $\mathcal{TIS}$;

2) there exists a cardinal $\exists\lambda \geq 2$ such that any topological Brandt λ-extension $\forall B_\lambda(S)$ of the semigroup *S* is *H*-closed in the $\mathcal{TIS}$ (*indeed there are many topological Brandt extensions – it depends of cardinal and of topology; moreover for each* λ *can exist more than two different topologies on* $B_\lambda(S)$);

3) for each cardinal $\forall\lambda \geq 2$ any topological Brandt λ-extension $\forall B_\lambda(S)$ of the semigroup *S* is *H*-closed in the $\mathcal{TIS}$.

Proof. Theorem 2 implies the implications 1) $\Rightarrow$2) and 1) $\Rightarrow$3). The implication 3) $\Rightarrow$2) is trivial.

We shall show that the implication 2)$\Rightarrow$1) holds. Suppose contrary ($2\wedge\neg 1$): there exists non *H*-closed topological inverse semigroup *S* in the $\mathcal{TIS}$ such, that for some $\lambda_0 \geq 2$ every topological Brandt λ_0-extension $B_\lambda(S)$ is *H*-closed in $\mathcal{TIS}$. Then there exists a topological inverse semigroup *T* which contains *S* as a non-closed subsemigroup. Let τ_T be a direct sum topology on $B_\lambda(T)$.Then, obviously, the semigroup $B_\lambda(S)$ with direct sum topology is not a closed subsemigroup of topological inverse semigroup $(B_\lambda(T), \tau_T)$, a contradiction with *H*-closedness of $B_\lambda(S)$. ◀

Analogous theorems 2 and 3 are true for algebraic closed inverse semigroups in $\mathcal{TIS}$.

An algebraic semigroup *S* is called *algebraically closed in* $\mathcal{S}$ if this semigroup with any semigroup topology is *H*-closed in $\mathcal{S}$ and $(S,\tau)\in\mathcal{S}$.

The same is true for absolute *H*-closedness: namely, for every cardinal $\lambda \geq 2$ any topological Brandt λ-extension of the absolutely *H*-closed topological semigroup *S* is absolutely *H*-closed semigroup in the class of topological inverse semigroups. The proof is much alike the proof for *H*-closedness case. And, as a consecuence for algebraic *h*-closed inverse semigroups in $\mathcal{TIS}$.

A semigroup is called *algebraically* h*-closed in* $\mathcal{S}$ if this semigroup with discrete topology δ is absolutely H–closed in $\mathcal{S}$ and $(S,\delta)\in\mathcal{S}$.

Now I'll present two examples which demonstrate, that in conditions 2) and 3) of Theorem 3 second quantum can not be replaced by "existence" $\exists$. This is a crucial moment – there are three equivalent conditions: *S* is an *H*-closed $\Leftrightarrow$ 2), and *S* is an *H*-closed $\Leftrightarrow$ 3); and nothing else, because it is proved that these conditions could not be weakened.

An example of on non *H*-closed topological inverse semigroup $\mathcal{N}$ in $\mathcal{TIS}$ is constructed, such that for arbitrary $\lambda \geq 2$ there exist H-closed topological Brandt λ-extension $B_\lambda(\mathcal{N})$, which implies that condition 2) in the above theorems could not be wakened neither to the: (*it is not sufficient condition for S to be H-closed*)

$2'$) $\exists\lambda \geq 2 \quad \exists B_\lambda(S)$ such that it is (absolutely) H–closed in $\mathcal{TIS}$.

Example I. $\mathcal{N} = (\mathbb{N}, \max, discr)$ - topological inverse semigroup.

We define topology τ_B on $B_2(\mathcal{N})$:

a) (α, x, β) isolated point in $B_2(\mathcal{N})$ for all $\alpha, \beta = 1, 2,\ x \in \mathcal{N}$;

б) family $\boldsymbol{\mathcal{B}}(0) = \{\{0\} \cup \{(\alpha, x, \beta)\,|\, \alpha, \beta = 1, 2,\, x \geq k\} \mid k \in \mathbb{N}\}$ is a base of τ_B in $0 \in B_2(\mathcal{N})$.

Obviously, $(B_2(\mathcal{N}), \tau_B)$ is a compact topological, thus *H*-closed semigroup. But $\mathcal{N}$ is not closed subsemigroup in $(B_2(\mathcal{N}), \tau_B)$. Thus, $\mathcal{N}$ is not *H*-closed in $\mathcal{TIS}$. ◀

An example of on non H-closed topological inverse semigroup $\mathcal{N}$ in $\mathcal{TIS}$ is constructed, such that for arbitrary $\lambda \geq 2$ there exist H-closed topological Brandt λ-extension $B_\lambda(\mathcal{N})$ in $\mathcal{TS}$, which implies that condition 3) in the above theorems could not be wakened to the:

$3'$) $\forall\lambda \geq 2 \quad \exists B_\lambda(S)$ such that it is (absolutely) H–closed in $\mathcal{TIS}$,

Example II. Let $\lambda \geq \omega$, and $\boldsymbol{\mathcal{N}} = (\mathbb{N}, \max, discr)$.

We define

$$V_\alpha(n) = B_\lambda(\mathcal{N}) \setminus \{(\alpha, k, \gamma) \mid \gamma \in I_\lambda,\ k < n,\ k \in \boldsymbol{\mathcal{N}}\},$$
$$H_\beta(n) = B_\lambda(\mathcal{N}) \setminus \{(\gamma, k, \beta) \mid \gamma \in I_\lambda,\ k < n,\ k \in \boldsymbol{\mathcal{N}}\}.$$

for any $\alpha, \beta \in I_\lambda$, $n \in \mathbb{N}$.

Put

$$U^{\alpha_1,\ldots,\alpha_l}(n) = \bigcap_{i=1}^{l} V_{\alpha_i}(n), \quad U_{\beta_1,\ldots,\beta_m}(n) = \bigcap_{j=1}^{m} H_{\beta_j}(n),$$
$$U^{\alpha_1,\ldots,\alpha_l}_{\beta_1,\ldots,\beta_m}(n) = U^{\alpha_1,\ldots,\alpha_l}(n) \cap U_{\beta_1,\ldots,\beta_m}(n),$$

where $\alpha_1,\ldots,\alpha_l, \beta_1,\ldots,\beta_m \in I_\lambda,\ l, m \in \mathbb{N},\ n \in \boldsymbol{\mathcal{N}}$. Then family

$$\boldsymbol{\mathcal{B}}^* = \{U^{\alpha_1,\ldots,\alpha_l}_{\beta_1,\ldots,\beta_m}(n) \mid \alpha_1,\ldots,\alpha_l, \beta_1,\ldots,\beta_m \in I_\lambda, n \in \boldsymbol{\mathcal{N}}\}$$
$$\bigcup \{(\alpha, s, \beta) \mid s \in \boldsymbol{\mathcal{N}}, \alpha, \beta \in I_\lambda\}$$

is a base of topology τ^* on $B_\lambda(\mathcal{N})$.◄

Proposition. For arbitrary $\lambda \geq \omega$, $(B_\lambda(\boldsymbol{\mathcal{N}}), \tau^*)$ is a topological *H*-closed semigroup. (*without proof*)

4. (Absolute) *H*-closedness of Brandt extension in the class of $\mathcal{TS}$: semigroup topologies on Brandt extensions which preserve *H*-closedness and an absolute *H*-closedness (constructions).

Previous theorems are not true for (absolutely) H-closed semigroups in $\mathcal{TS}$.

Since infinite discrete semigroup of matrix unitsis not *H*-closed (see example), topological Brandt λ-extensions do not preserve *H*-closedness (in $\mathcal{TS}$) for $\lambda \geq \omega$.

Example: countable B_ω with the discrete topology which is not H-closed in the class of locally compact topological semigroups.

Let $B_\omega = I_\omega \times I_\omega \bigcup \{0\}$ be the semigroup of matrix units and $a \notin B_\omega$.

Let $S = B_\omega \bigcup a$.

We put $aa = a0 = 0a = a(\alpha, \beta) = (\alpha, \beta)a = 0$ for all $(\alpha, \beta) \in B_\omega \setminus \{0\}$.

Further we enumerate the elements of the set I_ω by natural numbers. Let $A_n = \{(2k-1, 2k) : k \geq n\} \subset I_\omega \times I_\omega$ for each $n \in \mathbb{N}$.

A topology τ on $S = B_\omega \bigcup a$ is defined as follows:

1) all points of B_ω are isolated in S;

2) $\mathcal{B}(a) = \{U_n(a) = \{a\} \bigcup A_n : n \in \mathbb{N}\}$ is the base of topology τ at the point $a \in S$.

Then (*verify continuity of multiplication*):

a) $\{(l, m)\} \cdot U_n(a) = U_n(a) \cdot \{(l, m)\} = \{0\}$ for all $(\alpha, \beta) \in B_\omega \setminus \{0\}$, $n \geq \max\{l, m\}$; (*always a sequence which "starts" higher could be taken*)

b) $U_n(a) \cdot U_n(a) = U_n(a) \cdot \{0\} = \{0\} \cdot U_n(a) = \{0\}$ for any $n \in \mathbb{N}$; (*due to a neighbourhood od $a \in S$; that is why it has no "neighbour"-points*);

c) $U_n(a)$ is a compact subset of S for each $n \in \mathbb{N}$. (*each neighbour of a is compact – conjugate sequence* [Eng3.1.23]).

Therefore (S,τ) is a locally compact topological semigroup, but B_ω is not a closed subset of (S,τ) (*because* $a \notin B_\omega$ *is a limit point,* B_ω *is dense in* (S,τ)).

That is why a question naturally aroused: ***whether exist semigroup topologies on Brandt λ-extensions which preserve H-closedness in the class of*** $\mathcal{TS}$ ***?***

The answer is yes – for every infinite cardinal λ, semigroup topologies on Brandt λ-extensions which preserve *H*-closedness and an absolute *H*-closedness was constructed.

Construction of topologies:

Let (S,τ) be a topological semigroup, λ be an infinite cardinal.

For each $\alpha,\beta \in I_\lambda$ we define

$$V_\alpha = B_\lambda(S) \setminus \{(\alpha,a,\gamma) \mid \gamma \in I_\lambda, a \in S^1\},$$
$$H_\beta = B_\lambda(S) \setminus \{(\gamma,a,\beta) \mid \gamma \in I_\lambda, a \in S^1\}.$$

Put

$$U^{\alpha_1,\ldots,\alpha_n} = \bigcap_{i=1}^{n} V_{\alpha_i}, \quad U_{\beta_1,\ldots,\beta_m} = \bigcap_{j=1}^{m} H_{\beta_j}, \quad U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} = U^{\alpha_1,\ldots,\alpha_n} \bigcap U_{\beta_1,\ldots,\beta_m},$$

where $\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m \in I_\lambda$, $m,n \in \mathbb{N}$.

Let $\mathcal{B}$ be a base of topology of topological semigroup (S,τ). Further we define the following families

$$\mathcal{B}_{mv} = \{U^{\alpha_1,\ldots,\alpha_n} \mid \alpha_1,\ldots,\alpha_n \in I_\lambda, n \in \mathbb{N}\} \bigcup \{(\alpha,V,\beta) \mid V \in \mathcal{B}, \alpha,\beta \in I_\lambda\},$$
$$\mathcal{B}_{mh} = \{U_{\beta_1,\ldots,\beta_m} \mid \beta_1,\ldots,\beta_m \in I_\lambda, m \in \mathbb{N}\} \bigcup \{(\alpha,V,\beta) \mid V \in \mathcal{B}, \alpha,\beta \in I_\lambda\},$$
$$\mathcal{B}_{mi} = \{U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} \mid \alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m \in I_\lambda, n,m \in \mathbb{N}\} \bigcup \{(\alpha,V,\beta) \mid V \in \mathcal{B}, \alpha,\beta \in I_\lambda\}.$$

Obviously, the conditions (BP1)–(BP3)[Eng] hold for this families, and hence they are bases on $B_\lambda(S)$ of topologies $\tau_{mv}(S)$, $\tau_{mh}(S)$ and $\tau_{mi}(S)$, respectively.

Proposition. Let λ be an infinite cardinal, (S,τ) be a topological semigroup. Then

1) $(B_\lambda(S),\tau_{mv}(S))$, $(B_\lambda(S),\tau_{mh}(S))$ are topological semigroups;

2) if (S,τ) topological inverse semigroup, then $(B_\lambda(S),\tau_{mi}(S))$ is topological inverse semigroup.

Proof. Obviously it is sufficient to verify continuity of multiplication and inversion only in zero of $B_\lambda(S)$. Let us verify case 1).

For arbitrary open neighbourhood of zero $U^{\alpha_1,\ldots,\alpha_n}$, $\alpha_1,\ldots,\alpha_n \in I_\lambda$, and for all $\alpha,\beta \in I_\lambda$ hold:

$$U^{\alpha_1,\ldots,\alpha_n} \cdot U^{\alpha_1,\ldots,\alpha_n} \subseteq U^{\alpha_1,\ldots,\alpha_n},$$
$$(\alpha,S^1,\beta) \cdot U^{\alpha_1,\ldots,\alpha_n,\beta} = \{0\} \subseteq U^{\alpha_1,\ldots,\alpha_n},$$
$$U^{\alpha_1,\ldots,\alpha_n} \cdot (\alpha,S^1,\beta) = \{0\} \bigcup \{(\gamma,s,\beta) \mid \gamma \in I_\lambda \setminus \{\alpha_1,\ldots,\alpha_n\}, s \in S\} \subseteq U^{\alpha_1,\ldots,\alpha_n}.$$

Thus $(B_\lambda(S),\tau_{mv}(S))$ is a topological semigroup.◄

Theorem. Let $\lambda \geq \omega$ and (S,τ) be an *H*-closed topological semigroup. Then $(B_\lambda(S),\tau_{mv}(S))$, $(B_\lambda(S),\tau_{mh}(S))$ and $(B_\lambda(S),\tau_{mi}(S))$ are *H*-closed topological semigroups (in $\mathcal{TS}$).

Proof. Let us show that $(B_\lambda(S),\tau_{mi}(S))$ is H-closed semigroup. Other proofs are analogous.

Suppose that topological semigroup $(B_\lambda(S), \tau_{mi}(S))$ is not a closed subset in some topological semigroup *T*, which contains it as a subsemigroup: $(B_\lambda(S), \tau_{mi}(S)) \subset (T, \tau)$. Then there exists $x \in cl_T(B_\lambda(S)) \setminus B_\lambda(S) \subseteq T$. Previously there was proved that zero of $B_\lambda(S)$ coincides with zero of $cl_T(B_\lambda(S))$, that is why $0x = x0 = 0$. Than for (by multiplication continuity):

$$\forall W(0) \subset T \ \exists U(0), V(0), V(x) \subset T: \ V(x)V(0) \subseteq U(0), \ V(0)V(x) \subseteq U(0)$$

$V(x) \cap V(0) = \varnothing$, $U(0) \cap V(x) = \varnothing$, (because of Hausdorff topology), and

$U(0), V(0) \subseteq W(0)$ (there exist base element in any open set)

We can suppose that $U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m} = U(0) \cap B_\lambda(S)$ for some $\alpha_i, \beta_j \in I_\lambda$ (since $0_S = 0_T$).

Since S^1 is *H*-closed topological semigroup, one of the following conditions holds:

1) the set $B_{i_0} = V(x) \cap \{(\alpha_{i_0}, s, \gamma) \mid s \in S^1, \gamma \in I_\lambda\}$ cross infinite number of sets $S_{\alpha\beta}$ for some $i_0 \in \{1, \ldots, n\}$;
2) the set $B^{j_0} = V(x) \cap \{(\gamma, s, \alpha_{j_0}) \mid s \in S^1, \gamma \in I_\lambda\}$ cross infinite number of sets $S_{\alpha\beta}$ for some $j_0 \in \{1, \ldots, m\}$;

1) Let the first case holds. We put

$$\Gamma_{i_0} = \{\gamma \in I_\lambda \mid \text{there exists } s \text{ in } S^1 \text{ such, that } (\alpha_{i_0}, s, \gamma) \in V(x)\}.$$

Then the set $\{(\gamma, s, \gamma) \mid \gamma \in \Gamma_{i_0}, s \in S^1\} \cap U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}$ contains an infinite number of sets $S_{\alpha\alpha}$ for $\alpha \in I_\lambda$ for any basic neighborhood $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k} \subseteq V(0)$ of base $\tau_{mi}(S)$ in zero, where $\delta_1, \ldots, \delta_k \in I_\lambda$.

Then

$$B_{i_0} \cdot U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k} \not\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}.$$

Thus we have a contradiction with $V(x) \cdot V(0) \subseteq U(0)$.

2) If the second case holds, we put

$$\Gamma^{j_0} = \{\gamma \in I_\lambda \mid \text{there exists } s \text{ in } S^1 \text{ such, that } (\gamma, s, \alpha_{j_0}) \in V(x)\}.$$

Then the set $\{(\gamma, s, \gamma) \mid \gamma \in \Gamma^{j_0}, s \in S^1\} \cap U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k}$ contains an infinite number of sets $S_{\alpha\alpha}$ for $\alpha \in I_\lambda$ for any basic neighborhood $U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k} \subseteq V(0)$ of base $\tau_{mi}(S)$ in zero, where $\delta_1, \ldots, \delta_k \in I_\lambda$.

Then

$$U^{\delta_1,\ldots,\delta_k}_{\delta_1,\ldots,\delta_k} \cdot B^{j_0} \not\subseteq U^{\alpha_1,\ldots,\alpha_n}_{\beta_1,\ldots,\beta_m}.$$

Thus we have a contradiction with $V(0) \cdot V(x) \subseteq U(0)$ and $(B_\lambda(S), \tau_{mi}(S))$ is *H*-closed. ◄

5. Sufficient conditions on topological Brandt extensions for preserving the (absolute) H-closedness.

Recently we found the sufficient conditions on topological Brandt λ-extensions for preserving the (absolute) *H*-closedness.

Theorem 5.1. Let $B_\lambda(S)$ be a topological Brandt λ-extension of topological monoid S. If semigroup S is є *H*-closed and band of $B_\lambda(S)$ is compact, then $B_\lambda(S)$ is an ***H*-closed** topological semigroup.

Proof. Suppose $B_\lambda(S)$ is not H-closed topological semigroup. Then there exists topological semigroup T, which contains $B_\lambda(S)$ as non-closed subsemigroup. Since closure of subsemigroup in a topological semigroup is subsemigroup [CHK,I,p.9]), without loss of generality we can assume that $B_\lambda(S)$ is a closed subsemigroup in T and $T \setminus B_\lambda(S) \neq \varnothing$. Zero of $B_\lambda(S)$ is a zero for T (lemma 1). Let $x \in T \setminus B_\lambda(S)$. Then $x \cdot 0 = 0 \cdot x = 0$. By proposition 1, subset $S_{i,j}$ is closed in T for all $i, j \in I_\lambda$. Thus each open neighborhood of x intersect infinite number of sets $S_{i,j}$, $i, j \in I_\lambda$.

Let $U(x)$ and $U(0)$ be open neighborhoods of x and 0 in T, respectively, such that $U(x) \bigcap U(0) = \varnothing$. Since $x \cdot 0 = 0 \cdot x = 0$, from the continuity of semigroup operation in T follows, that there exist open neighborhoods $V(x)$ and $V(0)$ of x and 0 in T, respectively such that the following conditions hold:

$$V(x) \cdot V(0) \subseteq U(0),\ V(0) \cdot V(x) \subseteq U(0),\ V(0) \subseteq U(0) \text{ i } V(x) \subseteq U(x).$$

The compactness of band $E(B_\lambda(S))$ implies, that for arbitrary neighborhood $W(0)$ of zero 0 in T of semigroup $B_\lambda(S)$, the set $A(W(0)) = \{(i, 1_s, j) \mid (i, 1_s, j) \notin W(0)\}$ is finite. Thus, since neighborhood $V(x)$ intersect infinite number of sets $S_{i,j}$, $i, j \in I_\lambda$, at least one of the following conditions holds:

$$(V(x) \cdot V(0)) \cap V(x) \neq \varnothing \quad \text{or} \quad (V(0) \cdot V(x)) \cap V(x) \neq \varnothing,$$

a contradictions with the choice of neighborhoods $U(x)$ and $U(0)$. ◄

Theorem 5.2. Let $B_\lambda(S)$ be a topological Brandt λ-extension of topological monoid S. If semigroup S is є absolutely *H*-closed and band of $B_\lambda(S)$ is compact, then $B_\lambda(S)$ is **absolutely *H*-closed** topological semigroup.

Proof. Let T be arbitrary topological semigroup and $h: B_\lambda(S) \to T$ be a continuous homomorphism. Then $h(B_\lambda(S))$ is topological Brandt λ-extension of some topological monoid M. Namely, by [GutRep,SF2010,80,3.2: *every non-trivial homomorphic image of* $B_\lambda(S)$ *is the Brandt λ-extension of some monoid with zero. Moreover,* $h(B_\lambda(S))$ *is isomorphic to the Brand λ-extension of the homomorphic image of the monoid* $S_{\alpha,\alpha}$ *under the homomorphism h for any* $\alpha \in I_\lambda$] subsemigroup $h(B_\lambda(S))$ in T is Brandt λ-extension of subsemigroup $M = h(S_{i,i})$ in T for some $i \in I_\lambda$. Since subsemigroup M in T is topological monoid, definition of topological Brandt extension imply that $h(B_\lambda(S))$ is a topological Brandt λ-extension of topological monoid $M = h(S_{i,i})$ in T, for some $i \in I_\lambda$. Since S is absolutely H-closed, $h(S_{i,i})$ is closed suset in T and by theorem 5.1 subsemigroup $h(B_\lambda(S))$ is closed in T. ◄

Proposition 5.3. Let $B_\lambda(S)$ be a topological Brandt λ-extension of topological monoid S, and $B_\lambda(S)$ is a subset of topological semigroup T. Then for any element $x = (i, s, j) \in B_\lambda(S)$, $i, j \in I_\lambda$, exists open neighborhood $U(x)$ of x in T such that $U(x) \cap B_\lambda(S) \subseteq S_{i,j} \setminus \{0\}$.

Proof. The continuity of semigroup operation in T imply that for arbitrary open neighborhood $W(x)$ of x in T, such that $0 \notin W(x)$ there exist open neighborhood $U(x)$ of x in T such that $(i, 1_S, i) \cdot U(x) \cdot (j, 1_S, j) \subseteq W(x)$. Then $U(x) \cap B_\lambda(S) \subseteq S_{i,j} \setminus \{0\}$, since if not then $0 \in (i, 1_S, i) \cdot U(x) \cdot (j, 1_S, j) \subseteq W(x)$. ◄

Theorem 5.4. Let $B_\lambda(S)$ be a topological Brandt λ-extension of topological inverse monoid S with compact band E(S) in $\mathcal{TIS}$. If $E(B_\lambda(S))$ is regular space and $B_\lambda(S)$ is *H*-closed topological semigroup, then band $E(B_\lambda(S))$ is compact.

Proof. Suppose on contrary: the band $E(B_\lambda(S))$ is not a compact subset in $B_\lambda(S)$. Then, since band $E(S)$ of semigroup S is compact, then by proposition 5.3, there exist open neighborhood $U(0)$ of zero 0 of semigroup $B_\lambda(S)$ in $E(B_\lambda(S))$ such, that set $E(B_\lambda(S))\setminus U(0)$ intersect infinite number of sets $E(S)_{i,i}$, $i \in I_\lambda$.

Since $E(B_\lambda(S))$ is regular, there exists open neighborhood $V(0)$ of zero in $E(B_\lambda(S))$ such that $\overline{V(0)} \subseteq U(0)$. Without loss of generality, we can assume that we have finite number of such sets, and enumerate them by natural numbers: $E(S)_{i,i}$, $i = 1,2,3,\ldots$. Put $A_i = E(S)_{i,i} \setminus \overline{V(0)}$. Then A_i is open subset in $E(B_\lambda(S))$ and $A_i \cap U(0) = \varnothing$ for all $i = 1,2,3,\ldots$.

Define the maps $\pi_1 : B_\lambda(S) \to E(B_\lambda(S))$ and $\pi_2 : B_\lambda(S) \to E(B_\lambda(S))$ by formulae $\pi_1(s) = s \cdot s^{-1}$ and $\pi_2(s) = s^{-1} \cdot s$. Since $B_\lambda(S)$ is topological inverse semigroup, π_1 and π_2 are continuous. For arbitrary natural n designate $Z_n = \pi_1^{-1}(A_{2n-1}) \cap \pi_2^{-1}(A_{2n})$ and $P_n = \bigcup_{k=n}^{\infty} Z_k$. The continuity of maps π_1 and π_2 imply $\pi_1^{-1}(V(0)) \cap \pi_2^{-1}(V(0))$ and P_n are open subsets in $B_\lambda(S)$ for arbitrary natural n an obviously $\pi_1^{-1}(V(0)) \cap \pi_2^{-1}(V(0)) \cap P_n = \varnothing$.

Let $x \notin B_\lambda(S)$. We continue semigroup operation from $B_\lambda(S)$ onto $T = B_\lambda(S) \cup \{x\}$: $x \cdot x = s \cdot x = x \cdot s = 0$ for all $s \in B_\lambda(S)$. Obviously, it is associative binary operation on T. Let τ_B be a topology on $B_\lambda(S)$. Topology τ_T on T we define:

1) base of topologies τ_B and τ_T coincides in every point $s \in B_\lambda(S)$;
2) family $\Im(0) = \{U_n(x) = \{x\} \cup P_n \mid n = 1,2,3,\ldots\}$ is a base of topology τ_T in point $x \in T$.

Since $\pi_1^{-1}(V(0)) \cap \pi_2^{-1}(V(0)) \cap P_n = \varnothing$ and for arbitrary subset $S_{i,j} \setminus \{0\}$ in $s \in B_\lambda(S)$ exists natural m such that $S_{i,j} \setminus \{0\} \cap P_m = \varnothing$, then T is a Hausdorff space. Also for any open neighborhood of zero $W(0) \subseteq V(0)$ we have that

$$W(0) \cdot U_n(x) = U_n(x) \cdot W(0) = U_n(x) \cdot U_n(x) = \{0\} \subseteq W(0)$$

and since for any subset $S_{i,j} \setminus \{0\}$ in $B_\lambda(S)$ exist natural k such, that $S_{i,j} \setminus \{0\} \cdot P_k = P_k \cdot S_{i,j} \setminus \{0\} = \{0\} \subseteq W(0)$, T is a topological semigroup, which contains $B_\lambda(S)$ as dense subsemigroup. The obtained contradiction implies $E(B_\lambda(S))$ is compact subset in $B_\lambda(S)$. ◄

A topological space X is called *0-dimensional* if X is a non-empty T_1-space and has a base consisting of open-and-closed sets. Clearly, every zero-dimensional space is a Tychonoff space.

Theorem 5.5. Topological inverse Brandt semigroup $B_\lambda(G)$ with *H*-closed maximal subgroup in the class of topological semigroups is *H*-closed topological semigroup **if and only** if the band of $B_\lambda(G)$ is compact.

Proof. ($\Rightarrow$) By lemma [GP,SF05,4: any nonzero element of semigroup of matrix units B_λ is isolated in the closure of B_λ in S] arbitrary non-zero idempotent of band $E(B_\lambda(G))$ is an isolated point in $E(B_\lambda(G))$, thus topological space $E(B_\lambda(G))$ is 0-dimensional, thus regular. The conditions of theorem5.4 hold.

($\Leftarrow$) from theorem 5.2.

6. Application of Brandt extension to describing next classes of topological semigrups:

- **compact 0-simple topological inverse semigroups;**

- ***Suschkewitsch***: any finite semigroup *S* contains a minimal ideal *K*; he also showed that *K* is completely simple semigroup and described the structure of finite simple semigroups;
- ***Rees***: generalized Suschkiewitsch theorem and showed that if a semigroup *S* contains a minimal ideal *K* then *K* is isomorphic to a Rees matrix semigroup over a group *G* with a regular sandwich matrix *P*. He also proved that any completely 0-simple semigroup is isomorphic to a Rees matrix semigroup over a *0*-group *G* with a regular sandwich matrix *P*.
- ***Wallace***: proved the topological analogue of the Suschkewitsch-Rees Theorem for compact topological semigroups: every compact topological semigroup contains a minimal ideal, which is topologically iomorphic to a topological pararoup (*that is topological matrix Rees semigroup* $\mathcal{M}[G;I,\Lambda;P]$ *over a topological group G with regular sandwich matrix P.*)
- ***Paalman-de-Miranda***: any 0-simple compact topological semigroup S is completely 0-simple, the zero of S is an isolated point in S and S\{0} is homeomophic to the topological product $X\times G\times Y$, where X,Y are compact topological spaces and G is homeomorphic to the underlying space of a maximal subgroup of S, contained in S\{0}.
- ***Owen***: if S is a locally compact completely simple topological semigroup, then S has a structure similar to a copmact simple topological semigroup.

Let S be compact 0-simple topological inverse semigroup. Then S is completely 0-simple semigroup. By Rees Theorem and the next proposition [Dys,2.3.1]: the subsemigroup of idempotents of 0-simple compact topological inverse semigroup is finite (*follows from H-closedness of the semigroup of matrix units and because of it congruence-free property*), there exist a finite cardinal λ such that the semigroup S is algebraically isomorphic to the Brandt λ-extension of some group G. Since by corollary [Dys,2.3.1], the band of the semigroup $B_\lambda(G)$ is finite, then for arbitrary idempotents $e,f\in S$, $H(e,f)$ and $H(e)$ are clopen subsets in *S*.

Let S be an inverse semigroup. For any $e,f\in E(S)$ we put $H(e,f)=\{x\in S\mid xx^{-1}=e, x^{-1}x=f\}$ and $H(e)=H(e,e)$. Obviously $H(e)$ is a subgroup in S, moreover $H(e)$ is a maximal subgroup with unity e.

Thus, we obtain ***structural theorem for compact 0-simple topological inverse semigroups***:

Theorem. Let S be a 0-simple compact topological inverse semigroup. Then there exists nonempty finite set I_λ of cardinlity потужності λ and compact topological group H such, that S is topologically isomorphic to topological Brandt λ-extension $B_\lambda(H)$ of group H in the class of $\mathcal{TIS}$. Moreover, S is homeomorphic to a finite topological sum of compact topological groups and a single point.

Let λ be a cardinal, H be a group, $B_\lambda(H)$ be a Brandt λ-extension of group H and B_λ be a semigroup of matrix units. Then, obviously the map $f: B_\lambda(H) \to B_\lambda$, such that $f(0) = 0,\ f((\alpha, g, \beta)) = (\alpha, \beta)$, is a group homomorphism. The from the previous theorem follows

Corollary. Compact congruence-free topological inverses semigroup with zero is isomorphic to the finite semigroup of matrix units.

- ***Gutik, Repovs*** [SF07] generalized this result – described the structure of 0-simple countably compact topological inverse semigroups and the structure of congruence-free countably compact topological inverses semigroups. The proved that a countably compact topological inverse semigroup cannot contain the bicyclic semigroup. Therefore every simple countably compact topological inverse semigroup is completely simple.

- **compact and countably compact primitive topological inverse semigroups;**

We described the structure of compact and countably compact primitive topological inverse semigroups and showed that any countably compact primitive topological inverse semigroup embeds into a compact primitive topological inverse semigroup.

An *idempotent* e of a semigroup S without zero (with zero) is called *primitive* if e is a minimal elemnt in E(S) (in E(S)\{0}).

A nontrivial inverses semigroup ia called a *primitive* inverses semigroup if all its nonzero idempotents are primitive.

A semigroup S is a primitive inverses semigroup if and only if S is an orthogonal sum of Brandt semigroups [Petrich, II.4.3]

Let S be a semigroup with zero and let there be given a system of subsemigroups $\{S_\alpha\}_{\alpha\in A}$ such that $S_\alpha \cap S_\beta = S_\alpha S_\beta = \varnothing$ if $\alpha \neq \beta$ and $S = \bigcup_{\alpha\in A} S_\alpha$. In such a case, S is an *orthogonal sum* (or 0-direct union) of semigroups S_α to be denoted by $S = \sum_{\alpha\in A} S_\alpha$.

The following theorem describes ***the structure of primitive countably compact topological inverse semigroups***:

Theorem. Every primitive countably compact topological inverse semigroup S is topologically isomorphic to an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of topological Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of countably compact topological groups G_i in the class of topological inverse semigroups for some finite cardinals $\lambda_i \geq 1$. Moreover the family

$$\mathcal{B}(0) = \{S \setminus (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup \ldots \cup B_{\lambda_{i_n}}(G_{i_n}))^* \mid i_1, i_2, \ldots, i_n \in \mathcal{A}, n \in \mathbb{N}\}$$

Determines a base of topology at zero 0. [BeGP,13]

Proof. By [Petrich,II.4.3] the semigroup S is an orthogonal sum of Brandt semigroups and hence S is an orthogonal sum $\sum_{i\in\mathcal{A}} B_{\lambda_i}(G_i)$ of Brandt λ_i-extensions $B_{\lambda_i}(G_i)$ of groups G_i. We fix any $i_0 \in \mathcal{A}$. Since S is a topological inverse semigroup, [EbSeld,II.2] implies that $B_{\lambda_i}(G_i)$ is a topological inverse smigroup. By [BeGuPa,10,12] $B_{\lambda_i}(G_i)$ is a closed subsemigoup of S and hence by [Eng,3.10.4], $B_{\lambda_i}(G_i)$ is a countably compact 0-simple topological inverse semigroup. Then, by [GRep,SF07,2], the semigroup $B_{\lambda_i}(G_i)$ is a topological Brandt λ_i-extension of countably compact topological group G_{i_0} in the class of $\mathcal{TIS}$ for some finite cardinal $\lambda_{i_0} \geq 1$. This completes the proof of the first assertion of the theorem.

Suppose on the contrary that $\mathcal{B}(0)$ is a base at zero 0 of S. Then, there exists an open neighbourhood $\mathcal{U}(0)$ of zero 0 such that

$$\mathcal{U}(0) \cup (B_{\lambda_{i_1}}(G_{i_1}) \cup B_{\lambda_{i_2}}(G_{i_2}) \cup ... \cup B_{\lambda_{i_n}}(G_{i_n})) \setminus \{0\} \neq S$$

for finitely many indexes $i_1, i_2, ..., i_n \in \mathcal{A}$. Therefore there exists an infinitely family $\mathcal{F}$ of non-zero disjoint $\mathcal{H}$-classes such that $H \not\subset \mathcal{U}(0)$ for all $H \in \mathcal{F}$. Let $\mathcal{F}_0$ be an infinite countable subfamily of $\mathcal{F}$. We put $W = \bigcup\{H : H \in \mathcal{F} \setminus \mathcal{F}_o\}$. Lemma 9 [BeGuPa10] implies that the family $\mathcal{C} = \{\mathcal{U}(0), W\} \cup \mathcal{F}_o$ is an open countable cover of S. Simple observations shows that the cover $\mathcal{C}$ does not contain a finite subcover. This contradict to the countable compactness of S. The obtained contradiction implies the last assertion of the theorem.

Since any maximal subgroup of a compact topological semigroup *T* is a compact subset in *T,* the theorem implies the analogous for compact semigroups.

- **pseudocompact completely 0-simple topological inverse semigroups.**

A topological space X is called *pseudocompact* if every locally finite collection of open subsets of X is finite. A Tychonoff topological space X is *pseudocompact* if and only if each continuous real-valued function on X is bounded.

A family $\{A_s\}_{s \in S}$ of subsets of topological space X is called *locally finite*, if for arbitrary element $x \in X$ exists open neighbourhood U of x, such that the set $\{s \in S : U \cap A_s \neq \varnothing\}$ is finite. Obviously, that every countably compact space is pseudocompact.

Theorem. Every completely 0-simple pseudocompact topological inverse semigoup S is topological Brandt λ-extension $B_\lambda(G)$ of pseudocompact topological group *G* for some finite cardinal λ in the class of topological inverse semigrups.

Proof. By Rees theorem semigroup S is algebraically isomorphic to the Brandt λ-extension $B_\lambda(G)$ of some group *G*. Since inversion in S is continuous, the maps $\varepsilon^+ : S \to E(S)$ and $\varepsilon^- : S \to E(S)$, defined by formulae $\varepsilon^+(x) = x \cdot x^{-1}$ and $\varepsilon^-(x) = x^{-1} \cdot x$, respectively, are continuous. Thus, since continuous image of pseudocompact space is pseudocompact [Eng, 3.10.24], the $E(S)$ is psudocompact space. By lemme [MSt09,1] pseudocompact topological semilattice $E(S)$ is homeomorphic to the one-point Alexandroff compactification of the discrete space X of cardinality $|E(S)|$, with zero 0 as the remainder.

Thus, every non-zero idempotent of the semilattice $E(S)$ is an isolated point in $E(S)$. Since mappings ε^+ and ε^- are continuous, each maximal subgroup

$$H(e) = \{x \in S \mid \varepsilon^+(x) = \varepsilon^-(x) = e\},$$

which contains non-zero idempotent e, is open-closed subset in S. Then by [Eng,3.10.E] $H(e)$ is pseudocompact space.

Since S is 0-bisimple, from Clifford theorem [ClPr,2.20] follows that any subspace

$$H(e,f) = \{x \in S \mid \varepsilon^+(x) = e,\ \varepsilon^-(x) = f\}$$

in S, where $e, f \in E(S) \setminus \{0\}$, is homeomorphic to subspace of subgroup $H(e)$. Continuity of ε^{+} and ε^{-} implies, that $H(e, f)$ is a pseudocompact closopen subspace in S.

Analogously, $H(e) = H(e, e)$ is open-closed, and thus pseudocompact subset in S. Thus

$$S \setminus \{0\} = \bigoplus_{e, f \in E(S) \setminus \{0\}} H(e, f).$$

By [Eng.3.10.25] semilattice $E(S)$ is finite. Thus, since S is topological inverses semigroup, S is topological Brandt λ-extension $B_{\lambda}(G)$ of pseudocompact topological group $G = H(e)$ for some finite cardinal λ in the class of topological inverse semigroups. ◄

7. Example of the absolutely H-closed topological semigroup S with an absolutely H-closed ideal *I* such that the Rees quotient-semigroup S/I is not a topological semigroup.

Let *S* be a semigroup and *I* be an ideal of *S*. Using *S* and *I* one can construct a new semigroup by collapsing *I* into a single element while the elements of *S* outside of *I* retain their identity. The new semigroup obtained in this way is called the *Rees quotient semigroup* of *S* modulo *I*.

Suppose we are given a topological space *X* and an equivalence relation *E* on the set *X*. Let us denote by *X/E* the set of all equivalence classes of *E* and by *q* the mapping of *X* to *X/E* assigning to the point $x \in X$ the equivalence class $[x] \in X / E$. Now, in looking for a good topology on X / E, it is reasonable to require q to be continuous. It turns out that in the class of all topologies on X / E that make q continuous there exists the finest one: this is the family of all sets U such that $q^{-1}(U)$ is open in X. This topology is called the *quotient topology*, the set X / E equipped with it is called the quotient space, and $q : X \to X / E$ is called the natural quotient mapping, or briefly, the natural mapping.

A question "*when the Rees-quotient semigroup of topological semigroup with quotient topology is a topological semigroup?*" was investigated by many specialists in the topological semigroup theory.

- ***Wallace***: Rees-quotient semigroup of compact topological semigroup modulo closed ideal is a topological semigroup (if S is a compact topological semigroup and ρ is a closed congruence on S, then S/ρ is a compact topological semigroup);
- ***Lawson, Madsion**, 1971*: generalized Wallace's result for locally compact σ-compact topological semigroups (if S is a locally compact σ-compact topological semigroup and ρ is a closed congruence on S, then S/ρ is a topological semigroup; as an immediate corollary of the theorem, we have a topological version of the Rees quotient semigroup);
- ***Gutik***: observed that Rees-quotient semigroup of topological semigroup modulo compact ideal is a topological semigroup;
- ***Hryniv***: Lawson-Madison result does not hold for locally compact topological semigroups (an example of a locally compact metrizable topological semigroup S with a closed ideal I such that S/I is not a topological semigroup);

Since *H*-closed and absolutely *H*-closed topological semigrous are close to compact semigroups in relation to theirs topological properties, a question naturally aroused: *whether Rees-quotient semigroup of absolute H-closed topological semigroup modulo absolute H-closed ideal is a topological semigroup*?

Using the topological Brandt λ-extension of topological semigroup an example was constructed which demonstrate that the results of Lawson and Madison do not extend onto absolutely *H*-closed topological semigroups.

That is, an example of the countable absolutely *H*-closed 0-dimensional metrizable inverse topological semigroup *S* with an absolutely *H*-closed ideal *I* such that the Rees quotient-semigroup *S/I* is not a topological semigroup.

Example. Let $\mathbb{N}$ be a set of positive integers, $\{x_n\}$ be an increasing sequence in $\mathbb{N}$.

Put $\mathbb{N}^* = \{0\} \cup \left\{ 1/n \mid n \in \mathbb{N} \right\}$.

We define the semilattice operation on $\mathbb{N}^*$ as follows: $ab = \min\{a,b\}$, where $a,b \in \mathbb{N}^*$. Obviously, 0 is the zero of $\mathbb{N}^*$. We put $U_n(0) = \{0\} \cup \left\{ 1/x_k \mid k \geq n \right\}$, $n \in \mathbb{N}$.

A topology τ on $\mathbb{N}^*$ is defined as follows:

1) all nonzero elements of $\mathbb{N}^*$ are isolated points in $\mathbb{N}^*$;

2) $\mathcal{B}(0) = \{U_n(0) \mid n \in \mathbb{N}\}$ is the base of topology τ at the point $0 \in \mathbb{N}^*$.

It is easy to see that $(\mathbb{N}^*, \tau)$ is a countable linearly ordered σ-compact locally compact metrizable topological semilattice and if $x_{k+1} > x_k + 1$ for any $k \in \mathbb{N}$, then $(\mathbb{N}^*, \tau)$ is a non-compact semilattice.

By Proposition 1 [GutRep, SF08, 1] $(\mathbb{N}^*, \tau)$ is an *H*-closed topological semilattice and hence by [GutRep, SF08, 3, each linearly ordered H-closed topological semilattice is absolutely H-closed] the semilattice $(\mathbb{N}^*, \tau)$ is an absolutely *H*-closed.

Theorem. Let $\lambda = \omega$. Then $(B_\lambda(\mathbb{N}^*), \tau_{mv}(\mathbb{N}^*))$ and $(B_\lambda(\mathbb{N}^*), \tau_{mh}(\mathbb{N}^*))$ are metrizable topological semigoups.

The set $\mathcal{J}(\mathbb{N}^*) = \{0\} \cup \{(\alpha, 0, \beta) \mid 0 \in \mathbb{N}^*, \alpha, \beta \in I_\lambda\}$ is an ideal of $B_\lambda(\mathbb{N}^*)$. By [Dys,3.2.6] semigroup $\mathcal{J}(\mathbb{N}^*)$ with topology τ_{mh} induced from $(B_\lambda(\mathbb{N}^*), \tau_{mh}(\mathbb{N}^*))$ is absolutely H-closed topological semigroup, thus $\mathcal{J}(\mathbb{N}^*)$ is a closed ideal of $(B_\lambda(\mathbb{N}^*), \tau_{mh}(\mathbb{N}^*))$

Theorem. Let $\lambda \geq \omega$ and $\{x_n\}$ be an increasing sequence in $\mathbb{N}$ such that $x_{k+1} > x_k + 1$ for any $k \in \mathbb{N}$, and $(\mathbb{N}^*, \tau)$ be previously defined topological semigroup. Then topological Rees quotient semigroups

$$(B_\lambda(\mathbb{N}^*), \tau_{mv}(\mathbb{N}^*)) / \mathcal{J}(\mathbb{N}^*) \text{ and } (B_\lambda(\mathbb{N}^*), \tau_{mh}(\mathbb{N}^*)) / \mathcal{J}(\mathbb{N}^*)$$

with quotient topologies are not topological semigroups.

8. Open problems.

Gutik and Repovs [SF, 2010] studied algebraic properties of the Brandt λ^0–extensions of monoids with zero and non-trivial homomorphisms between the Brandt λ^0-extensions of monoids with zero. They also described a category whose objects are ingredients in the constructions of finite (compact, countably compact) topological Brandt λ^0-extensions of topological monoids with zeros.

- to investigate other topological properties of the Brandt λ^0-extension, the preserving of the minimality especially and discovering the minimal semigroup topologies on it. (Let S be minimal topological (inverse) semigroup. Whether exist on the $B_\lambda(S)$ minimal semigroup (inverse) topology such, that it induce on $S_{\alpha\alpha}$ the initial one? Under which conditions?
- to consider Rees matrix semigroups (not over grops) over Clifford semigroups, investigate its categorical description;
- to establish or, at least, to approach the criteria on different classes of topological semigroups to be H-closed (absolutely H-closed), thus solving one of the main problems in the theory of topological semigroups. On that way, we plan to solve Stepp Problem about H-closed semilattices in some classes of topological semilattices. One of the most important goals is to find new classes of H-closed semigroups. For that purpose we need to investigate the H-closedness and absolute H-closedness properties of the semigroup of finite partial bijection of a bounded rank I_λ^n as topological inverse semigroups.